Focusing and Polarization in Intuitionistic Logic

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Abstract. A focused proof system provides a normal form to cut-free proofs that structures the application of invertible and non-invertible inference rules. The focused proof system of Andreoli for linear logic has been applied to both the *proof search* and the *proof normalization* approaches to computation. Various proof systems in literature exhibit characteristics of focusing to one degree or another. We present a new, focused proof system for intuitionistic logic, called *LJF*, and show how other proof systems can be mapped into the new system by inserting logical connectives that prematurely stop focusing. We also use *LJF* to design a focused proof system for classical logic. Our approach to the design and analysis of these systems is based on the completeness of focusing in linear logic and on the notion of polarity that appears in Girard's LC and LU proof systems.

1 Introduction

Cut-elimination provides an important normal form for sequent calculus proofs. But what normal forms can we uncover about the structure of cut-free proofs? Since cut-free proofs play important roles in the foundations of computation, such normal forms might find a range of applications in the proof normalization foundations for functional programming or in the proof search foundations of logic programming.

1.1 About focusing

Andreoli's *focusing* proof system for linear logic (the *triadic* proof system of [1]) provides a normal form for cut-free proofs in linear logic. Although we describe this system, here called *LLF,* in more detail in Section 2, we highlight two aspect of focusing proofs here. First, linear logic connectives can be divided into the *asynchronous* connectives, whose right-introduction rules are invertible, and the *synchronous* connectives, whose right introduction rules are not (generally) invertible. The search for a focused proof can capitalize on this classification by applying (reading inference rules from conclusion to premise) all invertible rules in any order (without the need for backtracking) and by applying a chain of invertible rules that focus on a given formula and its positive subformulas. Such a chain of applications, usually called a *focus*, terminates when it reaches an asynchronous formula. Proof search can then alternate between applications of asynchronous introduction rules and chains of synchronous introduction rules.

A second aspect of focusing proofs is that the synchronous/asynchronous classification of non-atomic formulas must be extended to atomic formulas. The arbitrary assignment of positive (synchronous) and negative (asynchronous) *bias* to atomic formulas can have a major impact on, not the existence of focused proofs, but the shape of focused proofs. For example, consider the Horn clause specification of the Fibonacci series:

$$\mathit{fib}(0,0) \wedge \mathit{fib}(1,1) \wedge \forall n \forall f \forall f'[\mathit{fib}(n,f) \wedge \mathit{fib}(n+1,f') \supset \mathit{fib}(n+2,f+f')].$$

If all atomic formulas are given negative bias, then the only focused proofs of $\mathit{fib}(n, f_n)$ are those that can be classified as "backward chaining" (the size of the smallest one being exponential in n). On the other hand, if all atomic formulas are given positive bias, then the only focused proofs are those that can be classified as "forward chaining" (the size of the smallest one being linear in n).

1.2 Results

The contributions of this paper are the following. First, we introduce in Section 5 a new focusing proof system *LJF* and show that it is sound and complete for intuitionistic logic. Notable features of *LJF* are that it allows for atoms of different bias and it contains two versions of conjunction: while these conjunctions are logically equivalent, they are affected by focusing differently. Second, in Section 6, we show how several other focusing proof systems can be captured in *LJF*, in the sense of *full completeness* (one-to-one correspondence between proofs in different systems). One should note that while there are many focusing proof systems for intuitionistic logic in the literature, we appear to be the first to provide a single (intuitionistic) framework for capturing many of them. Third, in Section 7, we use *LJF* to derive *LKF*, a focusing system for classical logic.

1.3 Methodology and Related work

There are a number of sequent calculus proof systems known to be complete for intuitionistic logic that exhibit characteristics of focusing. Some of these proof systems are based on fixing globally on either forward chaining or backward chaining. The early work on *uniform proofs* [17] and the *LJT* proof system [11] are both backward chaining calculi (all atoms have negative bias). The *LJQ* calculus [11, 7] similarly selects the global preference to be forward chaining (all atoms have positive bias). Less has been published about systems that allow for mixing bias on atoms. The λRCC proof system of Jagadeesan, Nadathur, and Saraswat [13] allows for both forward chaining and backward chaining in a superset of the hereditary Harrop fragment of intuitionistic logic. Chaudhuri, Pfenning, and Price in [3] observed that focusing proofs with mixed biases on atoms can form a declarative basis for mixing forward and backward chaining within Horn clauses. The PhD theses of Howe [12] and Chaudhuri [2] also explored various focusing proof systems for linear and intuitionistic logic.

We are interested in providing a flexible and unifying framework that can collect together important aspects of many of these proof systems. There are several ways to motivate and validate the design of such a system. One approach stays entirely within intuitionistic logic and works directly with invertibility and permutability of inference rules. Such an approach has been taken in many papers, such as [17, 18, 7]. Our approach uses linear logic, with its exponential operators ! and ?, as a unifying framework for looking at intuitionistic (and classical) logic. The fact that Andreoli's focused system was defined for full linear logic provides us with a convenient platform for exploring the issues around focusing and polarity. We translate intuitionistic logic into linear logic, then show that proof systems for intuitionistic logic match focused proofs of the translated image (Section 3). A crucial aspect of understanding focusing in intuitionistic logic is provided by identifying the precise relationship between Andreoli's notion of polarity with Girard's notion of polarity found in the LC [9] and LU [10] systems (Section 4).

Another system concerning polarity and focusing is found in the work of Danos, Joinet and Schellinx [5, 6]. Many techniques that they developed, such as *inductive decorations,* are used throughout our analysis. Our work diverges from theirs in the adaptation of Andreoli's system (LLF) as our main instrument of construction. The LK_p^η system of [6] describes focused proofs for classical logic. Its connections to polarization and focusing were further explored and extended by Laurent, Quatrini and de Falco [14] using *polarized proof nets.* It may be tempting to speculate that the best way to arrive at a notion of intuitionistic focusing is by simple modifications to these systems, such as restricting them to single-conclusion sequents. Closer examination however, reveal intricate issues concerning this approach. For example, the notion of classical *polarity* appears to be distinct from and *contrary* to intuitionistic polarity, especially at the level of atoms (see Sections 4 and 7). Resolving this issue would be central to finding systems that support combined forward and backward-chaining. Although the relationship between LK_p^η and our systems is interesting, we chose for this work to derive intuitionistic focusing from focusing in linear logic as opposed to classical logic.

Much of the research into focusing systems has been motivated by their application. For example, the papers [13, 17, 12, 2] are motivated by foundational issues in logic programming and automated deduction. The papers [11, 5, 6, 14] are motivated by foundational issues in function programming and the λ-calculus. Also, Levy [15] presents focus-style proof systems for typing in the λ-calculus and Curien and Herbelin [4] (among others) have noted the relationship between forward chaining and call-by-value evaluation and between backward chaining and call-by-name evaluation.

Our work can be extended to second order logic, although this paper is concerned mainly with first-order intuitionistic logic.

The details missing from this paper can be found in the report [16].

2 Focusing in Linear Logic

We summarize the key results from [1] on focusing proofs for linear logic.

A *literal* is either an atomic formula or the linear negation of an atomic formula. A linear logic formula is in *negation normal form* if it does not contain occurrences of $\multimap$ and if all negations have atomic scope. If K is literal, then $K^\perp$ denotes its complement: in particular, if K is $A^\perp$ then $K^\perp$ is A.

Connectives in linear logic are either *asynchronous* or *synchronous*. The asynchronous connectives are $\bot$, $⅋$, $?$, $\top$, $\&$, and $\forall$ while the synchronous connectives are their de Morgan dual, namely, $\mathbf{1}$, $\otimes$, $!$, $\mathbf{0}$, $\oplus$, and $\exists$. Asynchronous connectives are those where the right-introduction rule is always invertible. Formally, a formula in negation normal form is of three kinds: literal, asynchronous (*i.e.*, its top-level connective is asynchronous), and synchronous (*i.e.*, its top-level connective is synchronous).

$$\frac{\Psi\colon \Delta \Uparrow L}{\Psi\colon \Delta \Uparrow \bot, L}\,[\bot] \qquad \frac{\Psi\colon \Delta \Uparrow F, G, L}{\Psi\colon \Delta \Uparrow F ⅋ G, L}\,[⅋] \qquad \frac{\Psi, F\colon \Delta \Uparrow L}{\Psi\colon \Delta \Uparrow ?F, L}\,[?]$$

$$\frac{}{\Psi\colon \Delta \Uparrow \top, L}\,[\top] \qquad \frac{\Psi\colon \Delta \Uparrow F, L \quad \Psi\colon \Delta \Uparrow G, L}{\Psi\colon \Delta \Uparrow F \,\&\, G, L}\,[\&] \qquad \frac{\Psi\colon \Delta \Uparrow B[y/x], L}{\Psi\colon \Delta \Uparrow \forall x.B, L}\,[\forall]$$

$$\frac{\Psi\colon \Delta, F \Uparrow L}{\Psi\colon \Delta \Uparrow F, L}\,[R\Uparrow] \quad \text{provided that F is not asynchronous}$$

$$\frac{}{\Psi\colon \cdot \Downarrow \mathbf{1}}\,[\mathbf{1}] \qquad \frac{\Psi\colon \Delta_1 \Downarrow F \quad \Psi\colon \Delta_2 \Downarrow G}{\Psi\colon \Delta_1, \Delta_2 \Downarrow F \otimes G}\,[\otimes] \qquad \frac{\Psi\colon \cdot \Uparrow F}{\Psi\colon \cdot \Downarrow !F}\,[!]$$

$$\frac{\Psi\colon \Delta \Downarrow F_1}{\Psi\colon \Delta \Downarrow F_1 \oplus F_2}\,[\oplus_l] \qquad \frac{\Psi\colon \Delta \Downarrow F_2}{\Psi\colon \Delta \Downarrow F_1 \oplus F_2}\,[\oplus_r] \qquad \frac{\Psi\colon \Delta \Downarrow B[t/x]}{\Psi\colon \Delta \Downarrow \exists x.B}\,[\exists]$$

$$\frac{\Psi\colon \Delta \Uparrow F}{\Psi\colon \Delta \Downarrow F}\,[R\Downarrow] \quad \text{provided that } F \text{ is either asynchronous or a negative literal}$$

$$\text{If } K \text{ a positive literal:} \quad \frac{}{\Psi\colon K^\perp \Downarrow K}\,[I_1] \qquad \frac{}{\Psi, K^\perp\colon \cdot \Downarrow K}\,[I_2]$$

$$\text{If } F \text{ is not a negative literal:} \quad \frac{\Psi\colon \Delta \Downarrow F}{\Psi\colon \Delta, F \Uparrow \cdot}\,[D_1] \qquad \frac{\Psi, F\colon \Delta \Downarrow F}{\Psi, F\colon \Delta \Uparrow \cdot}\,[D_2]$$

Fig. 1. The focused proof system LLF for linear logic

The focusing proof system LLF for linear logic, presented in Figure 1, contains two kinds of sequents. In the sequent $\Psi\colon \Delta \Uparrow L$, the "zones" Ψ and Δ are multisets and L is a list. This sequent encodes the usual one-sided sequent $\vdash ?\Psi, \Delta, L$ (here, we assume the natural coercion of lists into multisets). This sequent will also satisfy the invariant that requires Δ to contain only literals and synchronous formulas. In the sequent $\Psi\colon \Delta \Downarrow F$, the zone Ψ is a multiset of formulas and Δ is a multiset of literals and synchronous formulas, and F is a single formula.

As mentioned in Section 1.1, the classification of non-atomic formulas as asynchronous or synchronous can naturally be pushed to literals: an atom linked

B	B^1	B^0	$(B^0)^\perp$
atom Q	Q	Q	$Q^\perp$
true	1	$\top$	0
false	0	0	$\top$
$P \wedge Q$	$!(P^1 \& Q^1)$	$!P^0 \& !Q^0$	$?(P^0)^\perp \oplus ?(Q^0)^\perp$
$P \vee Q$	$!P^1 \oplus !Q^1$	$!P^0 \oplus !Q^0$	$?(P^0)^\perp \& ?(Q^0)^\perp$
$P \supset Q$	$!(?(P^0)^\perp \parr Q^1)$	$!P^1 \multimap !Q^0$	$!P^1 \otimes ?(Q^0)^\perp$
$\neg P$	$!(0 \parr ?(P^0)^\perp)$	$!P^1 \multimap 0$	$!P^1 \otimes \top$
$\exists x P$	$\exists x\, !P^1$	$\exists x\, !P^0$	$\forall x\, ?(P^0)^\perp$
$\forall x P$	$!\forall x P^1$	$\forall x\, !P^0$	$\exists x\, ?(P^0)^\perp$

Table 1. The 0/1 translation used to encode LJ proofs into linear logic.

to asynchronous behavior is said to have a *negative bias* while an atom linked to synchronous behavior is said to have a *positive bias*. In Andreoli's original presentation of LLF [1] all atoms were classified as "positive" and their negations "negative." Such a choice was also made by Girard for LC [9]. In a classical setting, such a choice works fine since classical negation simply flips bias. In intuitionistic systems, however, a more natural treatment is to assign an arbitrary bias directly to atoms: before one can speak of a proof in LLF, the atoms must be divided into two sets: one for *positive* atoms and the other for *negative* atoms. This bias of atoms is extended to literals: negating a negative atom yields a positive literal and negating a positive atom yields a negative literal. Notice that the bias of literals is explicitly referred to in the $[R \Uparrow]$ and initial rules: in particular, in the initial rules, the literal on the right of the $\Downarrow$ must be positive.

Changes to the bias assigned to atoms does not affect provability of a linear logic formula: instead it affects the structure of focused proofs.

3 Translating Intuitionistic Logic

Table 1 contains a translation of intuitionistic logic into linear logic. This translation induces a bijection between arbitrary LJ proofs and LLF proofs of the translated image in the following sense. First notice that this translation is *asymmetric*: the intuitionistic formula A is translated using A^1 if it occurs on the right-side of an LJ sequent and as A^0 if it occurs on the left-side. Since this translation is used to capture cut-free proofs, such distinctions are not problematic. Since the left-hand side of a sequent in LJ will be negated when translated to a one-sided linear logic sequent, $(B^0)^\perp$ is also shown. The following Proposition essentially says that, via this translation, linear logic focusing can capture arbitrary proofs in LJ. Here, $\vdash_I$ denotes an intuitionistic logic sequent.

Proposition 1. *Let $(\Gamma^0)^\perp$ be the multiset $\{(D^0)^\perp \mid D \in \Gamma\}$. The focused proofs of $\vdash (\Gamma^0)^\perp :\Uparrow R^1$ are in bijective correspondence with the proofs of $\Gamma \vdash_I R$.*

For a detailed proof, see [16]. We simply illustrate one case in constructing the mapping from a collection of LLF rules to an LJ inference rule.

$$\dfrac{\dfrac{\dfrac{\vdash (\Gamma^0)^\perp, (D_i^0)^\perp : R^1 \Uparrow}{\vdash (\Gamma^0)^\perp : R^1 \Uparrow ?(D_i^0)^\perp}\,[?]}{\vdash (\Gamma^0)^\perp : R^1 \Downarrow ?(D_i^0)^\perp}\,[R\Downarrow]}{\vdash (\Gamma^0)^\perp : R^1 \Downarrow ?(D_1^0)^\perp \oplus ?(D_2^0)^\perp}\,[\oplus] \qquad \longmapsto \qquad \dfrac{\Gamma, D_i \vdash_I R}{\Gamma, D_1 \wedge D_2 \vdash_I R}\,[\wedge L]$$

The liberal use of ! in this translation *throttles* focusing. This translation is reminiscent of the earliest embedding of classical into intuitionistic logic of Kolmogorov, which uses the double negation in a similarly liberal fashion.

The 0/1 translation can be used as a starting point in establishing the completeness of other proof systems. These systems can be seen as *induced* from alternative translations of intuitionistic logic. Consider, for example, the LJQ' proof system presented in [7]. The translation for this system is given here using the "q/j" mapping. The "$\otimes 1$" device is another way to control focusing, or the lack thereof. *All atoms must be given positive bias* for this translation.

F	F^q (right)	F^j (left)
atom C	C	C
false	0	0
$A \wedge B$	$A^q \otimes B^q$	$!\,A^j \otimes !\,B^j$
$A \vee B$	$A^q \oplus B^q$	$!\,A^j \oplus !\,B^j$
$A \supset B$	$(!\,A^j \multimap B^q) \otimes 1$	$A^q \multimap !\,B^j$

With minor changes, Girard's original (non-polarized) translation of intuitionistic logic [8] induces the complement to LJQ' called LJT [11], which is itself derived from LKT [5] (where this connection was noted implicitly.) *All atoms must be given negative bias* for this translation.

Given a translation such as that of LJQ', one can give a completeness proof for the system using a *"grand tour"* through linear logic as follows:

1. Show that a proof under the 0/1 translation can be converted into a proof under the new translation. This usually follows from cut-elimination.
2. Define a mapping between proofs in the new system (such as LJQ) and LLF proofs of its translation.
3. Show soundness of the new system with respect to LJ. This is usually trivial. The "tour" is now complete, since proofs in LJ map to proofs under the 0/1 translation.

An intuitionistic system that contains atoms of both positive and negative bias is λRCC [13]. Two special cases of the $\supset L$ rule are distinguished involving $E \supset D$ for positive atom E and $G \supset A$ for negative atom A. Each rule requires that the complementary atom (E on the left, A on the right) is present when applied, thus terminating one branch of the proof. One can translate these special cases using forms $E \multimap !\,D'$ and $!\,G' \multimap A$, respectively, in linear logic. The strategy outlined above can then be used to not only prove its completeness but also extend it with more aggressive focusing features.

Our interest here is not the construction of individual systems but the building of a unifying framework for focusing in intuitionistic logic. Such a task requires a closer examination of *polarity* and its connection to focusing.

4 Permeable Formulas and their Polarity

Focused proofs in linear logic are characterized by two different phases: the invertible (asynchronous) phase and the non-invertible (synchronous) phase. These two phases are characterized by introduction rules for dual sets of formulas. In order to construct a general focusing scheme for intuitionistic logic, the non-linear (exponential) aspects of proofs need special attention, especially in light of the fact that the [!] rule stops a bottom-up construction of focused application of synchronous rules (the arrow $\Downarrow$ in the conclusion flips to $\Uparrow$ in the premise).

For our purposes here, a particularly flexible way to deal with the exponentials in the translations of intuitionistic formulas is via the notion of *permeation* that is used in LU [10]. In particular, there are essentially three grades of *permeation.* The formula B is *left-permeable* if $B \equiv\, !\, B$, is *right-permeable* if $B \equiv\, ?\, B$, and *neutral* otherwise. Within sequent calculus proofs, a formula is left-permeable if it admits structural rules on the left and right-permeable if it admits structural rules on the right. An example of a left-permeable formula is $\exists x\, !A$. All left-permeable formulas are synchronous and all right-permeables asynchronous. In the LU system, both the left and right sides of sequents contain two zones — one that treats formulas linearly and one that permits structural rules. A left-permeable (resp., right-permeable) formula is allowed to move between both zones on the left (right). In addition, LU introduces atoms that are inherently left or right-permeable or neutral. Although they appear to properly extend linear logic, one can simulate LU in "regular" linear logic by translating left-permeable atoms A as $!\, A$ and right-permeable ones as $?\, A$.

To preserve the focusing characteristics of permeable atoms as positively or negatively biased atoms, we use the following LU-inspired asymmetrical translation. The superscript -1 indicates the left-side translation and $+1$ indicates the right-side translation:

$P^{-1} = !\, P$ and $P^{+1} = P$, for left-permeable (positive) atom P.
$N^{-1} = N$ and $N^{+1} = ?\, N$, for right-permeable (negative) atom N.
$B^{-1} = B^{+1} = B$, for neutral atom B.

The ! rule of LLF causes a loss of focus in all circumstances, and is the main reason why we use an asymmetrical translation. The translation of positive atoms above preserves *permeation on the left* while allowing for *focus on the right.* That is, left-permeable atoms can now be interpreted meaningfully as positively biased atoms in focused proofs, and dually for right-permeable atoms. Furthermore, the permeation of positive atoms is *"one-way only:"* they cannot be selected for focus again once they enter the non-linear context.

Intuitionistic logic uses the left-permeable and neutral formulas and atoms. LU defines a translation for intuitionistic logic so that *all synchronous formulas* *are left-permeable.* For example, $\vee$ is translated as follows (here, P, Q are positive and N, M are negative): $(P \vee Q)^{-1} = P^{-1} \oplus Q^{-1}$, $(P \vee N)^{-1} = P^{-1} \oplus {!}N^{-1}$, $(N \vee P)^{-1} = {!}N^{-1} \oplus Q^{-1}$, and $(N \vee M)^{-1} = {!}N^{-1} \oplus {!}M^{-1}$. The final element of intuitionistic polarity is that *neutral atoms should be assigned negative bias in focused proofs.* Neutral atoms that are introduced into the left context (e.g. by a $\supset L$ rule) must immediately end that branch of the proof in an identity rule. Otherwise, the unique *stoup* is lost when multiple non-permeable atoms accumulate in the linear context.

The LU and LLF systems serve as a convenient platform for the unified characterization of polarity and focusing in all three logics. We can now understand the terminology of "positive" and "negative" formulas in each logic as follows:

Linear logic: *Positive* formulas are synchronous formulas and positively biased neutral atoms. *Negative* formulas are asynchronous formulas and negatively biased neutral atoms.

Intuitionistic logic: *Positive* formulas are left-permeable formulas and atoms. *Negative* formulas are asynchronous neutral formulas and negatively biased neutral atoms.

Classical logic: *Positive* formulas are left-permeable formulas and atoms. *Negative* formulas are right-permeable formulas and atoms.

5 The *LJF* Sequent Calculus

Since the polarities of intuitionistic logic observe stronger invariances, intuitionistic focused proofs are more well-structured than LLF proofs. The non-linear context of LLF contains both synchronous and asynchronous formulas, whereas in intuitionistic logic sequents can be clearly divided into zones respecting polarity. That is, when translating an intuitionistic sequent into a LLF sequent, synchronous formulas on the left are placed in the linear context.

We also make an adjustment on the LU translation of intuitionistic logic. Instead of using & or $\otimes$ depending on the polarities of the subformulas, we construct two versions of intuitionistic conjunction, which has the following meaning in linear logic (P, Q for positives, N, M for negatives, A, B arbitrary):

$$
\begin{array}{ll}
(P \wedge^+ Q)^{-1} = P^{-1} \otimes Q^{-1} & (A \wedge^+ B)^{+1} = A^{+1} \otimes B^{+1} \\
(P \wedge^+ N)^{-1} = P^{-1} \otimes {!}N^{-1} & \\
(N \wedge^+ P)^{-1} = {!}N^{-1} \otimes P^{-1} & (A \wedge^- B)^{-1} = A^{-1} \& B^{-1} \\
(N \wedge^+ M)^{-1} = {!}N^{-1} \otimes {!}M^{-1} & (A \wedge^- B)^{+1} = A^{+1} \& B^{+1}
\end{array}
$$

The connectives $\wedge^-$ and $\wedge^+$ are equivalent in intuitionistic logic in terms of provability but differ in their impact on the structure of focused proofs. The use of two conjunctions means that the top-level structure of formulas completely determines their polarity. *Polarity* in intuitionistic logic is defined as follows

Definition 1. *Atoms in LJF are arbitrarily positive or negative.* Positive formulas *are among positive atoms, true, false, $A \wedge^+ B$, $A \vee B$ and $\exists x A$.* Negative formulas *are among negative atoms, $A \wedge^- B$, $A \supset B$ and $\forall x A$.*

$$\frac{[N,\Gamma] \xrightarrow{N} [R]}{[N,\Gamma] \longrightarrow [R]}\ Lf \qquad \frac{[\Gamma] -_{P}\rightarrow}{[\Gamma] \longrightarrow [P]}\ Rf \qquad \frac{[\Gamma], P \longrightarrow [R]}{[\Gamma] \xrightarrow{P} [R]}\ R_l \qquad \frac{[\Gamma] \longrightarrow N}{[\Gamma] -_{N}\rightarrow}\ R_r$$

$$\frac{[C,\Gamma],\Theta \longrightarrow \mathcal{R}}{[\Gamma],\Theta, C \longrightarrow \mathcal{R}}\ []_l \qquad \frac{[\Gamma],\Theta \longrightarrow [D]}{[\Gamma],\Theta \longrightarrow D}\ []_r$$

$$\frac{}{[P,\Gamma] -_{P}\rightarrow}\ I_r,\ \text{atomic } P \qquad \frac{}{[\Gamma] \xrightarrow{N} [N]}\ I_l,\ \text{atomic } N$$

$$\frac{}{[\Gamma],\Theta, false \longrightarrow \mathcal{R}}\ falseL \qquad \frac{[\Gamma],\Theta \longrightarrow \mathcal{R}}{[\Gamma],\Theta, true \longrightarrow \mathcal{R}}\ trueL \qquad \frac{}{[\Gamma] -_{true}\rightarrow}\ trueR$$

$$\frac{[\Gamma] \xrightarrow{A_i} [R]}{[\Gamma] \xrightarrow{A_1 \wedge^- A_2} [R]}\ \wedge^- L \qquad \frac{[\Gamma],\Theta, A, B \longrightarrow \mathcal{R}}{[\Gamma],\Theta, A\ \wedge^+ B \longrightarrow \mathcal{R}}\ \wedge^+ L$$

$$\frac{[\Gamma],\Theta \longrightarrow A \quad [\Gamma],\Theta \longrightarrow B}{[\Gamma],\Theta \longrightarrow A\ \wedge^- B}\ \wedge^- R \qquad \frac{[\Gamma] -_{A}\rightarrow \quad [\Gamma] -_{B}\rightarrow}{[\Gamma] -_{A\ \wedge^+ B}\rightarrow}\ \wedge^+ R$$

$$\frac{[\Gamma],\Theta, A \longrightarrow \mathcal{R} \quad [\Gamma],\Theta, B \longrightarrow \mathcal{R}}{[\Gamma],\Theta, A \vee B \longrightarrow \mathcal{R}}\ \vee L \qquad \frac{[\Gamma] -_{A_i}\rightarrow}{[\Gamma] -_{A_1 \vee A_2}\rightarrow}\ \vee R$$

$$\frac{[\Gamma] -_{A}\rightarrow \qquad [\Gamma] \xrightarrow{B} [R]}{[\Gamma] \xrightarrow{A \supset B} [R]}\ \supset L \qquad \frac{[\Gamma],\Theta, A \longrightarrow B}{[\Gamma],\Theta \longrightarrow A \supset B}\ \supset R$$

$$\frac{[\Gamma],\Theta, A \longrightarrow \mathcal{R}}{[\Gamma],\Theta, \exists y A \longrightarrow \mathcal{R}}\ \exists L \qquad \frac{[\Gamma] -_{A[t/x]}\rightarrow}{[\Gamma] -_{\exists x A}\rightarrow}\ \exists R \qquad \frac{[\Gamma] \xrightarrow{A[t/x]} [R]}{[\Gamma] \xrightarrow{\forall x A} [R]}\ \forall L \qquad \frac{[\Gamma],\Theta \longrightarrow A}{[\Gamma],\Theta \longrightarrow \forall y A}\ \forall R$$

Fig. 2. The Intuitionistic Sequent Calculus LJF. Here, P is positive, N is negative, C is a negative formula or positive atom, and D a positive formula or negative atom. Other formulas are arbitrary. Also, y is not free in Γ, Θ, or R.

The above translation induces the sequent calculus *LJF* for intuitionistic logic, shown in Figure 2. Sequents in *LJF* can be interpreted as follows:

1. $[\Gamma],\Theta \longrightarrow \mathcal{R}$ (end sequent): this is an *unfocused sequent.* Γ contains negative formulas and positive atoms.
2. $[\Gamma] \longrightarrow [R]$: this represents a sequent in which all asynchronous formulas have been decomposed, and is ready for a formula to be selected for focus.
3. $[\Gamma] \xrightarrow{A} [R]$: this is a *left-focusing* sequent, with focus on formula A. The meaning of this sequent remains $\Gamma, A \vdash_I R$.
4. $[\Gamma] -_{A}\rightarrow$: this is a *right-focusing* sequent on formula A, with the meaning $\Gamma \vdash_I A$.

Theorem 1. *LJF is sound and complete with respect to intuitionistic logic.*

Proof. Using the "grand tour" strategy. See [16, Section 6] for details.

Given the different forms of sequents, the cut rule for *LJF* takes many forms:

$$\frac{[\Gamma],\Theta \longrightarrow P \quad [\Gamma'],\Theta',P \longrightarrow \mathcal{R}}{[\Gamma\Gamma'],\Theta\Theta' \longrightarrow \mathcal{R}}\ Cut^{+} \qquad \frac{[\Gamma],\Theta \longrightarrow C \quad [C,\Gamma'],\Theta' \longrightarrow \mathcal{R}}{[\Gamma\Gamma'],\Theta\Theta' \longrightarrow \mathcal{R}}\ Cut^{-}$$

$$\frac{[\Gamma] \xrightarrow{B} [P] \quad [\Gamma'],P \longrightarrow [R]}{[\Gamma\Gamma'] \xrightarrow{B} [R]}\ Cut_1^{\leftarrow} \qquad \frac{[\Gamma] \longrightarrow N \quad [N,\Gamma'] \xrightarrow{B} [R]}{[\Gamma\Gamma'] \xrightarrow{B} [R]}\ Cut_2^{\leftarrow}$$

$$\frac{[\Gamma] -_{C}\!\!\rightarrow \quad [C,\Gamma'] -_{R}\!\!\rightarrow}{[\Gamma\Gamma'] -_{R}\!\!\rightarrow}\ Cut^{\rightarrow}$$

Notice that the last three cut rules retain focus in the conclusion. These rules extend those of *LJQ'* [7], which were shown to be useful for studying term-reduction systems. See [16] for a proof of the admissibility of these rules.

Like LLF, a key characteristic of *LJF* is the assignment of arbitrary polarity to atoms. To illustrate the effect of these assignments on the structure of focused proofs, consider the sequent $a, a \supset b, b \supset c \vdash c$ where a, b and c are atoms. This sequent can be proved either by *forward chaining* through the clause $a \supset b$, or *backward chaining* through the clause $b \supset c$. Assume that atoms a and b are assigned positive polarity and that c is assigned negative polarity. This assignment effectively adopts the forward chaining strategy, reflected in the following *LJF* proof segment (here, Γ is the set $\{a, a \supset b, b \supset c\}$):

$$\frac{\dfrac{}{[\Gamma] -_{a}\!\!\rightarrow}\ I_r \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[b,\Gamma] -_{b}\!\!\rightarrow}\ I_r \quad \dfrac{}{[b,\Gamma] \xrightarrow{c} [c]}\ I_l}{[b,\Gamma] \xrightarrow{b\supset c} [c]}\ \supset L}{[b,\Gamma] \longrightarrow [c]}\ Lf}{[\Gamma], b \longrightarrow [c]}\ []_l}{[\Gamma] \xrightarrow{b} [c]}\ R_l}{[\Gamma] \xrightarrow{a\supset b} [c]}\ \supset L$$

The polarities of a and c do not fundamentally affect the structure of the proof in this example. However, assigning negative polarity to atom b would restrict the proof to use the backchaining strategy:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[\Gamma] -_{a}\!\!\rightarrow}\ I_r \quad \dfrac{}{[\Gamma] \xrightarrow{b} [b]}\ I_l}{[\Gamma] \xrightarrow{a\supset b} [b]}\ \supset L}{[\Gamma] \longrightarrow [b]}\ Lf}{[\Gamma] \longrightarrow b}\ []_r}{[\Gamma] -_{b}\!\!\rightarrow}\ R_r \qquad \dfrac{}{[\Gamma] \xrightarrow{c} [c]}\ I_l}{[\Gamma] \xrightarrow{b\supset c} [c]}\ \supset L$$

6 Embedding Intuitionistic Systems in LJF

The *LJF* proof system can be used to "host" other focusing proof system for intuitionistic logic. One obvious restriction to *LJF* is its purely negative fragment, which essentially corresponds to LJT. In the negative fragment one also finds *uniform proofs,* where the right "goal" formula is always fully decomposed before any left rule is applied. Various other proof systems can be embedded into *LJF* by mapping intuitionistic formulas to intuitionistic formulas in such a way that focusing features in *LJF* are stopped by the insertion of *delay* operators. In particular, if we define $\partial^-(B) = true \supset B$ and $\partial^+(B) = true \wedge^+ B$, then B, $\partial^-(B)$, and $\partial^+(B)$ are all logically equivalent but $\partial^-(B)$ is always negative and $\partial^+(B)$ is always positive.

Proofs in the LJQ' system can be embedded into *LJF* by translating all left-side formulas (l) as negatives and all right-side formulas (r) as positives: in particular, for atom B, $B^l = B^r = B$, $false^l = \partial^-(false)$, $false^r = false$, $(A \wedge B)^l = \partial^-(A^l \wedge^+ B^l)$, $(A \wedge B)^r = A^r \wedge^+ B^r$, $(A \vee B)^l = \partial^-(A^l \vee B^l)$, $(A \vee B)^r = A^r \vee B^r$, $(A \supset B)^l = A^r \supset \partial^+(B^l)$, $(A \supset B)^r = \partial^+(A^l \supset B^r)$.

Arbitrary LJ proofs can be embedded within *LJF* be inserting sufficient delaying operators. The table here provides the translation (redefining the superscripts l and r, for convenience). Together with cut-elimination, the embedding also suggests a completeness proof for *LJF* independently of linear logic. The following example embeds the $\wedge R$ rule in *LJF*:

F	F^l (left)	F^r (right)
atom C	C	C
false	$\partial^-(false)$	*false*
true	$\partial^-(true)$	*true*
$A \wedge B$	$\partial^+(A^l) \wedge^- \partial^+(B^l)$	$\partial^+(A^r \wedge^- B^r)$
$A \vee B$	$\partial^-(A^l \vee B^l)$	$\partial^-(A^r) \vee \partial^-(B^r)$
$A \supset B$	$\partial^-(A^r) \supset \partial^+(B^l)$	$\partial^+(A^l \supset B^r)$
$\exists x A$	$\partial^-(\exists x A^l)$	$\exists x \partial^-(A^r)$
$\forall x A$	$\forall x \partial^+(A^l)$	$\partial^+(\forall x A^r)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\Gamma] \longrightarrow [A^r]}{[\Gamma] \longrightarrow A^r}\,[]_r \quad \dfrac{[\Gamma] \longrightarrow [B^r]}{[\Gamma] \longrightarrow B^r}\,[]_r}{[\Gamma] \longrightarrow A^r \wedge^- B^r}\,\wedge^- R}{[\Gamma] -_{A^r \wedge^- B^r}\!\rightarrow}\,R_r \qquad \dfrac{}{[\Gamma] -_{true}\!\rightarrow}\,trueR}{[\Gamma] -_{(A^r \wedge^- B^r) \wedge^+ true}\!\rightarrow}\,\wedge^+ R}{[\Gamma] \longrightarrow [\partial^+(A^r \wedge^- B^r)]}\,Rf$$

The system λRCC also presents interesting choices. In particular, it may not always be the best choice to focus maximally. Forward chaining may generate a new formula or "clause" that may need to be used multiple times. In a $\supset L$ rule on formulas $E \supset D$ where E is a positive atom, one may not wish to decompose the formula D immediately. This is accomplished in the linear translation with a !. It can also be accomplished by using formulas $E \supset \partial^+(D)$ in case D is negative, and $E \supset \partial^+(\partial^-(D))$ in case D is positive. Note that unlike the l/r translations for LJQ and LJ above, these simple devices do not hereditarily alter the structure of D.

7 Embedding Classical Logic in LJF

We can use *LJF* to formulate a focused sequent calculus for classical logic that reveals the latter's constructive content in the style of LC. While it is possible to derive such a system again using linear logic, classical logic can also be embedded within intuitionistic logic using the *double-negation* translations of Gödel, Gentzen, and Kolmogorov. These translations do not, however, yield significant focusing features. Girard's *polarized* version of the double negation translation for LC approaches the problem of capturing duality in a more subtle way. Following the style of *LJF*, we wish to define dual versions of each propositional connective, which leads to a more usable calculus. We thus modify the LC translation in a natural way, which is consistent with its original intent. The proof system we derive is called *LKF*.

We must first separate classical from intuitionistic polarity since these are different notions (see the end of Section 4).

Definition 2. *Atoms are arbitrarily classified as either positive or negative. The literal* $\neg A$ *has the opposite polarity of the atom* A. Positive formulas *are among positive literals,* $\mathcal{T}$, $\mathcal{F}$, $A \wedge^+ B$, $A \vee^+ B$, $A \supset^+ B$ *and* $\exists x A$. Negative formulas *are among negative literals,* $\neg\mathcal{T}$, $\neg\mathcal{F}$, $A \wedge^- B$, $A \vee^- B$, $A \supset^- B$ *and* $\forall x A$. *Negation* $\neg A$ *is defined by de Morgan dualities* $\neg A/A$, $\wedge^+/\vee^-$, $\wedge^-/\vee^+$ *and* $\forall/\exists$. *Negative implication* $A \supset^- B$ *is defined as* $\neg A \vee^- B$ *and* $A \supset^+ B$ *is defined as* $\neg A \vee^+ B$. *Formulas are assumed to be in negation normal form (that is, formulas that do not contain implications and where negations have atomic scope).*

The constants $\mathcal{T}$, $\mathcal{F}$, $\neg\mathcal{T}$ and $\neg\mathcal{F}$ are best described, respectively, as 1, 0, $\bot$ and $\top$ in linear logic. Just as we have dual versions of each connective, we also have dual versions of each identity. But this is not linear logic as the formulas are polarized *in the extreme*. The distinction between the positive and negative versions of each connective affects only the structure of proofs and not provability.

Let $\sim A$ represent the intuitionistic formula $A \supset \phi$ where ϕ is *some unspecified positive atom.* The "$\approx$" embedding of classical logic is found in Table 2. Variations are possible on the embedding. Note that the classical $\wedge^-$ is not defined in terms of the intuitionistic $\wedge^-$. The embeddings are selected to enforce the dualities $\wedge^-/\vee^+$ and $\wedge^+/\vee^-$. Alternatives may also work, but will increase the complexity of the derivation. Here, the cases all follow the pattern P or $\sim P$ where P is a positive intuitionistic formula. In particular, negative intuitionistic atoms are not used in the embedding.

The $\approx$ embedding induces the *LKF* sequent calculus in Figure 3 from the image of *LJF* proofs, analogous to how *LJF* was derived from LLF. Here is one sample correspondence between a *LJF* rule and a LKF rule:

$$\frac{[\Delta], \Psi, A, B \longrightarrow [\phi]}{[\Delta], \Psi, A \wedge^+ B \longrightarrow [\phi]}\ \wedge^+ L \qquad \longmapsto \qquad \frac{\vdash [\Theta], \Gamma, A, B}{\vdash [\Theta], \Gamma, A \vee^- B}\ \vee^-$$

Sequents of the form $\vdash [\Theta], \Gamma$ are unfocused while those of the form $\mapsto [\Theta], A$ focus on the *stoup* formula A.

$\mathcal{A}^{\approx}$	$\mathcal{B}^{\approx}$	$(\mathcal{A} \wedge^{+} \mathcal{B})^{\approx}$	$(\mathcal{A} \wedge^{-} \mathcal{B})^{\approx}$	$(\mathcal{A} \vee^{+} \mathcal{B})^{\approx}$	$(\mathcal{A} \vee^{-} \mathcal{B})^{\approx}$	$(\neg\mathcal{A})^{\approx}$
A	B	$A \wedge^{+} B$	$\sim(\sim A \vee \sim B)$	$A \vee B$	$\sim(\sim A \wedge^{+} \sim B)$	$\sim A$
A	$\sim B$	$A \wedge^{+} \sim B$	$\sim(\sim A \vee B)$	$A \vee \sim B$	$\sim(\sim A \wedge^{+} B)$	.
$\sim A$	B	$\sim A \wedge^{+} B$	$\sim(A \vee \sim B)$	$\sim A \vee B$	$\sim(A \wedge^{+} \sim B)$	A
$\sim A$	$\sim B$	$\sim A \wedge^{+} \sim B$	$\sim(A \vee B)$	$\sim A \vee \sim B$	$\sim(A \wedge^{+} B)$	.

$\mathcal{A}^{\approx}$	$\mathcal{B}^{\approx}$	$(\mathcal{A} \supset^{+} \mathcal{B})^{\approx}$	$(\mathcal{A} \supset^{-} \mathcal{B})^{\approx}$	$(\forall x\mathcal{A})^{\approx}$	$(\exists x\mathcal{A})^{\approx}$
A	B	$\sim A \vee B$	$\sim(A \wedge^{+} \sim B)$	$\sim(\exists x \sim A)$	$\exists x A$
A	$\sim B$	$\sim A \vee \sim B$	$\sim(A \wedge^{+} B)$	.	.
$\sim A$	B	$A \vee B$	$\sim(\sim A \wedge^{+} \sim B)$	$\sim(\exists x A)$	$\exists x \sim A$
$\sim A$	$\sim B$	$A \vee \sim B$	$\sim(\sim A \wedge^{+} B)$	.	.

Table 2. Polarized embedding of classical logic. The $(\cdot)^{\approx}$ translation on compound formulas is given above (there, A, B represent formulas not preceded by $\sim$). For positive classical atom P, $P^{\approx} = P$; for negative classical atom N, $N^{\approx} = \sim N$; (where both P and N are assigned positive intuitionistic polarity), and for the logical constants $\mathcal{T}^{\approx} = \mathit{true}$, $\mathcal{F}^{\approx} = \mathit{false}$, $(\neg\mathcal{T})^{\approx} = \sim \mathit{true}$, $(\neg\mathcal{F})^{\approx} = \sim \mathit{false}$.

The following correctness theorem for LKF can be proved by relating it to the Gödel-Gentzen translation (see [16, Section 9] for more details).

Theorem 2. *LKF is sound and complete with respect to classical logic.*

We have constructed this embedding of classical logic as a further demonstration of the abilities of *LJF* as a hosting framework. The embedding also revealed interesting relationships between classical and intuitionistic polarity. It is also possible to derive LKF from linear logic: one would only need to define each connective to be either wholly positive or negative. For example, the translation of $(A \vee^{-} B)^{p}$ is $A^{p} \parr B^{p}$ if A^{p} and B^{p} are both negative; is $A^{p} \parr\ ?\,B^{p}$ if only A^{p} is negative; is $?\,A^{p} \parr B^{p}$ if only B^{p} is negative; and is $?A^{p} \parr ?B^{p}$, if A^{p} and B^{p} are both positive. This translation is called the *"polaro"* translation in [6], where it was used to formulate LK^{η}_{p}, the first focused proof system for classical logic. Like the $\approx$ translation, the polaro translation is a derivative of the LC/LU analysis of polarity. LKF is derivable from LLF using (essentially) the polaro translation in the same manner that *LJF* is derived.

LK^{η}_{p} was extended to $LK^{\eta,\rho}_{pol}$ in [14]. These systems were formulated independently of Andreoli's results. The authors of [6] opted not to present LK^{η}_{p} as a sequent calculus because they feared that it will have the cumbersome size of LU. Such cumbersomeness can, in fact, be avoided by adopting LLF-style *reaction* rules.

Given our goals, the choice in adopting Andreoli's system is justified in that LKF and LJF have the form of compact sequent calculi ready for implementation. More significantly perhaps, LK^{η}_{p} and $LK^{\eta,\rho}_{pol}$ define focusing for classical logic. They map to polarized forms of linear logic (LLP and LL_{pol}). LLF is defined for full classical linear logic. LKF is embedded within LLF in the same way that LC is embedded within LU. LLF is well suited for hosting other logics.

$$\frac{\vdash [\Theta, C], \Gamma}{\vdash [\Theta], \Gamma, C} [] \qquad \frac{\mapsto [P, \Theta], P}{\vdash [P, \Theta]} \; Focus \qquad \frac{\vdash [\Theta], N}{\mapsto [\Theta], N} \; Release$$

$$\frac{}{\mapsto [\neg P, \Theta], P} \; ID^{+}, \text{ atomic } P \qquad \frac{}{\mapsto [N, \Theta], \neg N} \; ID^{-}, \text{ atomic } N$$

$$\frac{}{\mapsto [\Theta], \mathcal{T}} \; indeed \qquad \frac{}{\vdash [\Theta], \Gamma, \neg \mathcal{F}} \; absurd \qquad \frac{\vdash [\Theta], \Gamma}{\vdash [\Theta], \Gamma, \neg \mathcal{T}} \; trivial$$

$$\frac{\vdash [\Theta], \Gamma, A \quad \vdash [\Theta], \Gamma, B}{\vdash [\Theta], \Gamma, A \wedge^{-} B} \; \wedge^{-} \qquad \frac{\vdash [\Theta], \Gamma, A, B}{\vdash [\Theta], \Gamma, A \vee^{-} B} \; \vee^{-}$$

$$\frac{\vdash [\Theta], \Gamma, B, \neg A}{\vdash [\Theta], \Gamma, A \supset^{-} B} \; \supset^{-} \qquad \frac{\vdash [\Theta], \Gamma, A}{\vdash [\Theta], \Gamma, \forall x A} \; \forall$$

$$\frac{\mapsto [\Theta], A \quad \mapsto [\Theta], B}{\mapsto [\Theta], A \wedge^{+} B} \; \wedge^{+} \qquad \frac{\mapsto [\Theta], A_i}{\mapsto [\Theta], A_1 \vee^{+} A_2} \; \vee^{+} \qquad \frac{\mapsto [\Theta], A[t/x]}{\mapsto [\Theta], \exists x A} \; \exists$$

$$\frac{\mapsto [\Theta], \neg A}{\mapsto [\Theta], A \supset^{+} B} \; \supset^{+} \qquad \frac{\mapsto [\Theta], B}{\mapsto [\Theta], A \supset^{+} B} \; \supset^{+}$$

Fig. 3. The Classical Sequent Calculus LKF. Here, P is positive, N is negative, C is a positive formula or a negative literal, Θ consists of positive formulas and negative literals, and x is not free in Θ, Γ. End-sequents have the form $\vdash [], \Gamma$.

8 Conclusion and Future Work

We have studied focused proof construction in intuitionistic logic. The key to this endeavor is the definition of polarity for intuitionistic logic. The *LJF* proof system captures focusing using this notion of polarity. We illustrate how systems such as LJ, LJT, LJQ, and λRCC can be captured within *LJF* by assigning polarity to atoms and by adding to intuitionistic logic formulas annotations on conjunctions and delaying operators. We also use *LJF* to derive and justify the *LKF* focusing proof system for classical logic.

It remains to examine the impact of these focusing calculi on typed λ-calculi, logic programming, and theorem proving. Given the connections observed between LJQ/LJT and call-by-name/value, the *LJF* system could provide a framework for λ-term evaluations that combine the eager and lazy evaluation strategies. In the area of theorem proving, there are a number of completeness theorems for various restrictions to resolution: it would be interesting to see if any of these are captured by an appropriate mapping into *LJF*.

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