A Unified Proof-Theoretic Framework for Classical, Intuitionistic and Linear Logics

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Abstract. We introduce a unified proof system *LUF* based on the concepts of polarization and focusing. Connectives from intuitionistic, classical and linear logic can mix with few restrictions in this system. The strongest form of completeness is achieved in that cut-elimination in the unified system immediately proves cut-elimination in each of these logics. This system is based on an expansion of the concept of *focusing* by introducing substages of the synchronous and asynchronous phases of proofs.

1 Introduction

In what sense can intuitionistic, classical and linear logics be seen as components of a unified system? One simple answer is that they are all fragments of classical logic, with different restrictions on how proofs can be composed, and with some extra symbols in the case of linear logic[1]. However, we wish to obtain from a unified logic the unity of certain properties such as cut-elimination. A cut-elimination proof for classical logic is not a proof of the same property in linear logic. Even in the case of intuitionistic logic, it has to be confirmed that the elimination procedure *stayed within the intuitionistic restriction.* Moreover, a unified logic should be a platform on which deduction in the different sublogics can *mix.* For example, one may wish to add to an intuitionistic system the linear implication symbol and write formulas such as $A \supset (B \multimap C)$. The utility of such a mixture has been demonstrated [9].

It is well-known that linear logic can embed both classical and intuitionistic logic. Yet linear logic also leaves questions unanswerd if considered as a unified logic. The problem here is the lack of *full adequacy* in the embedding. Once a classical/intuitionistic formula is translated into a linear formula, it becomes subject to a wider range of proof rules, not all of which correspond to valid inferences in classical/intuitionistic logic. In particular, in embedding the intuitionistic sequent calculus, the linear logic translation would not enforce the "single-conclusion" invariant. For example, in the *implication-left* rule, which is interpreted by the $\otimes$ rule, one might end up with two formulas on the right-hand side in one of the premises[2]. Even if we are only concerned with *successful proofs,* there are still more such proofs in linear logic then there are in the logic that's being embedded (e.g., when 0 or $\top$ is involved, such as in $0 \vdash A, B$). For the same

[1] If we considered both $\otimes$ and $\&$ as $\wedge$, both $\invamp$ and $\oplus$ as $\vee$, and ignored all occurrences of ? and !, then every linear logic proof is clearly a classical proof.

[2] Full adequacy may be achieved by extending linear logic with *subexponentials* (see [2]).

reasons, the cut-elimination procedure for linear logic is not immediately such a procedure for intuitionistic and classical logics. Linear logic will still have an important role in this paper as a tool for analysis.

The main contributions of this paper are as follows:

- A unified system in which connectives of classical, intuitionistic, and multiplicative-additive linear logics can mix with few restrictions.
- A *fully adequate* proof-theoretic embedding of classical and intuitionistic logics (as well as other variants). In particular, the cut-elimination proof of the unified sequent calculus is at once also a cut-elimination proof of each of its fragments.
- An expansion of the concept of *focusing* as defined by Andreoli [1]. In particular, we provide a unified sequent calculus that refines the *synchronous* and *asynchronous* phases of proofs into additional sub-stages.

In this unified system it would be possible, for example, to develop a single theorem prover that can be applied immediately to each of its fragments: classical, intuitionistic or otherwise.

Proof theoretically, the core distinction between classical, intuitionistic and linear logics can be described by the role of structural rules. In this paper, the *polarity* of a formula refers to *the relationship between the formula and the applicability of structural rules.* In Girard's LC system [6], it was recognized that different polarities exist even in classical logic, and that this allows for classical proofs to be interpreted constructively. By assigning different polarities to different versions of the connectives, one is able to influence the structure of proofs. A more aggressive use of polarization is found in the unified logic LU [7]. This logic is close in character to the one we shall define here. The polarity concept of LC and LU have impacted several subsequent studies such as [3, 12], as well our own work [13, 15]. Another concept of "polarization" can be found in the *focused* proofs of Andreoli [1] as the duality between *synchronous* and *asynchronous* inference rules. The utility of focusing has been amply demonstrated. If we expand the notion of "structural rule" to include more than just contraction and weakening, then focusing can also be seen as a form of polarization in the sense defined here.

However, we find that even this expanded analysis of polarization is incomplete: *there are still not enough polarities.* The problem is in how intuitionistic logic interact with the polarities of classical and linear logic. In LU, some intuitionistic polarities are equated with linear polarities (LU is unfocused so they're referred to as "neutrals"). In LU, classical, intuitionistic and linear logic exist as fragments and can interact through cut-elimination where their well-formed formulas intersect. However, the ability to use "hybrid" formulas such as $A \parr (B \supset C)$ in LU is limited because cut-elimination is only available in certain contexts, a limitation due to the fact that there are insufficient polarity distinctions to determine how to permute cuts in all cases (see Section A). The new dimension of polarization that we introduce here, called left-versus-right, is a generalization of the familiar intuitionistic principle due to Gentzen (see also [10]). However, instead of distinguishing between left- and right-hand *sides* of sequents, *left* and *right* become polarity attributes attached to formulas. Along with the polarities identified by Girard and by Andreoli, we define a system based on three axes of polarization corresponding to three layers of focusing. Each axes holds a significant logic, namely classical logic, multiplicative-additive linear logic (MALL), and the "purely negative" fragment of intuitionistic logic (with implication and universal quantification). In this context "full" intuitionistic logic, with disjunction and existential quantification, is already seen as a hybrid system. We will complete the integration of these polarities into a unified system called LUF . However, LUF is much more than a focused version of LU: there are fundamental differences in the set of connectives and their polarities, especially in terms of cut-elimination. This unity is also much more than a disjoint union of three known logics. New kinds of formulas can be written. Other fragments of LUF are also identified that can stand on their own as new logics.

2 Synthetic Connectives and Focusing

To explain the role of focusing in our logic, we will need to expand the scope of focusing beyond the original concept introduced by Andreoli. A core property of focused proofs is that they enable the formulation of *synthetic logical connectives.* In particular, linear logic, with its binary connectives $\oplus, \otimes, \parr, \&$, their units, and the exponentials $!, ?$, provides a rich framework for studying this concept.

To what extent can connectives be synthesized into new ones? As remarked by Girard [8], introduction rules for synthetic connectives should support *initial elimination,* i.e., the principle that initial sequents $\vdash A, A^{\perp}$ (or $A \vdash A$) can be derived from atomic instances of initial sequents. To see how this principle can fail, consider the combination $\otimes\&$ as a possible synthetic connective: ***e.g.,*** the "synthetic introduction rule" for the formula $A \otimes (B \,\&\, C)$ should result from combining the usual rules for $\otimes$ and $\&$ as:

$$\frac{\vdash A, \Delta_1 \quad \vdash B, \Delta_2 \quad \vdash C, \Delta_2}{\vdash A \otimes (B \,\&\, C), \Delta_1 \Delta_2} \otimes\&$$

We must also consider its dual, $A \parr (B \oplus C)$, for which the following introduction rules are immediate:

$$\frac{\vdash A, B, \Delta}{\vdash A \parr (B \oplus C), \Delta} \parr\oplus \qquad \frac{\vdash A, C, \Delta}{\vdash A \parr (B \oplus C), \Delta} \parr\oplus$$

These rules are clearly sound, and in fact cut-elimination is preserved: one can check that a cut between the introduced formulas can be reduced to cuts on their sub-formulas. But initial-elimination fails: one cannot prove $\vdash A \otimes (B \,\&\, C), A^{\perp} \parr (B^{\perp} \oplus C^{\perp})$ using *these* introduction rules.

The four combinations of $\otimes\&$, $\otimes\parr$, $\oplus\&$ and $\oplus\parr$ (and their duals) all fail the initial-elimination test. The *focused* proof system of Andreoli [1] provides a recipe for the formulation of valid synthetic connectives, with limitations. In that system, which we refer to as *LLF,* connectives are divided into *negatives* (invertible right-introduction rules) $\parr, \&, \forall$ and *positives* (non-invertible right-introduction rules) $\oplus, \otimes, \exists$. These sets of connectives are De Morgan duals of each other and any collection of connectives of the *same* polarity forms a proper synthetic connective. We *focus* on a positive (synchronous) formula and maintain that focus on its immediate positive subformulas: focus is then broken when negative (asynchronous) subformulas are encountered.

Although Andreoli's notion of focusing allows the formulation of some synthetic connectives, it does not extend completely to include structural rules, which in linear logic are enabled with $?$ and $!$. Significantly, the initial elimination test also shows why forms such as $!(A \oplus B)$ and $?(A \parr B)$ (i.e., a ! before a positive formula or a ? before a negative formula) cannot be considered synthetic connectives. Also failing the test are forms such as $?!(A \& B)$ and $!?(A \oplus B)$. However, combinations such as $?(A \oplus B)$ and $!\forall x.A$ do form valid synthetics. Other formulas, such as $!(?A \parr !B)$, yield synthetics *up to a point* as long as some exponential operators are preserved in the premises of the introduction rules. The synthetic introduction rules for $!(?A \parr !B)$ and its dual can be

$$\frac{\vdash ?A, B, ?\Delta}{\vdash !(?A \parr !B), ?\Delta} \qquad \frac{\vdash !A, ?\Delta \quad \vdash B, \Delta'}{\vdash ?(!A \otimes ?B), ?\Delta, \Delta'}$$

where $?\Delta$ represents a multiset of ?-formulas.

How does the relationship between focusing and synthetic connectives extend to this setting? Consider, for example, the formula $A \oplus ?(B \oplus C)$. Clearly focusing must stop before the ? because a structural rule may be needed. Expectedly, this formula would fail the initial elimination test if the two $\oplus$-introductions are combined into one. But now consider the formula $?(A \oplus ?(B \oplus C))$. This combination *does* form a valid synthetic along with its dual. Yet, in LLF, this formula remains unfocusable because ? is simply classified as "negative."

One might observe that $?(A \oplus ?(B \oplus C))$ is logically equivalent to $?(A \oplus (B \oplus C))$, which is focusable past the ?. But in a context where we wish to *combine* connectives from different logics, dropping an exponential operator has consequences: *is the inner $\oplus$ supposed to represent an intuitionistic disjunction, or a classical one?* The consequence is the loss of *full adequacy* in the encoding of these logics. In particular, if ! is dropped from $!A \multimap B$, then intuitionistic proofs may loose the single-conclusion property.

The above example shows that what defines focusing should be expanded beyond the distinction between "positives" and "negatives." Finer distinctions are needed.

3 Polarities and Connectives

In this section we define the syntax of LUF formulas. One of the first important uses of "polarity" can be attributed to Gentzen in defining LJ. We have chosen to name our *left* and *right* polarities in a way that is consistent with intuitionistic logic. We now introduce the logical connectives of LUF along with their *polarity classifications*.

Positive Left: $\boxplus$, $\boxtimes$, $\exists^l$, 0_l, positive-left literals. "$+L$" polarity.
Positive Right: $\vee^+$, $\wedge^+$, $\exists$, 0_r, 1_r, literals. "$+R$" polarity.
Positive Linear: $\otimes$, $\oplus$, Σ, 0, 1, literals. "$+1$" polarity.
Negative Linear: $\parr$, &, Π, $\top$, $\bot$, literals. "-1" polarity.
Negative Left: $\vee^-$, $\wedge^-$, $\forall$, $\bot_l$, $\top_l$, literals. "$-L$" polarity.
Negative Right: $\sqcup$, $\sqcap$, $\forall^r$, $\top_r$, literals. "$-R$" polarity.

The term *literal* above refers to an atomic formula A or its negation $A^\perp$. All formulas are written in negation normal form. The negation of non-atomic formulas is defined by the following De Morgan duals: $\otimes/\parr$, $\oplus/\&$, Σ/Π, $1/\bot$, $0/\top$, $\boxplus/\sqcap$, $\boxtimes/\sqcup$, $\exists^l/\forall^r$, $0_l/\top_r$, $\vee^+/\wedge^-$, $\wedge^+/\vee^-$, $\exists/\forall$, $0_r/\top_l$, $1_r/\bot_l$.

The six poles form three axes of polarization by De Morgan negation. The polarization scheme is also illustrated in Figure 1. The polarity of a formula is determined entirely by its top-level connective. For example, $A \parr B$, when both A and B are left-formulas, is still considered to be -1, although it is provably equivalent to $A \vee^- B$.

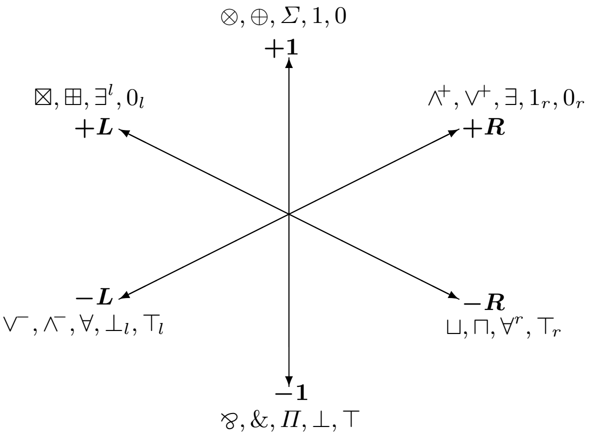

Fig. 1. Polarization in LUF

We have included three sets of constants or “units,” one on each axis, for the convenience of having constants of every polarity. $\top_l$, $\bot_l$, 1_r and 0_r are equivalent to their $+1$/-1 counterparts. $\top_r$ is an asynchronous version of 1 and 0_l a synchronous version of $\bot$. There is no 1_l or $\bot_r$: they would correspond to the connectives $\boxtimes$ and $\sqcup$ which are generalized forms of intuitionistic implication.

Restrictions on Formulas. The expressiveness of LUF is in allowing connectives of different polarities to mix with few restrictions. However, a design goal of LUF is that *contraction on asynchronous formulas is never required* Thus, we will remove formulas that resemble the forms $?(A \parr B)$ and $!(A \oplus B)$, since these not only compromise focusing but can also wreck havoc with cut-elimination. To achieve this invariant, formulas are restricted so that whenever they contain a subformula of the form $A \vee^- B$, $A \wedge^- B$ or $\forall x.A$, where A (or B) is negative then A (resp., B) must have the polarity $-L$. Dually, for $A \wedge^+ B$, $A \vee^+ B$ and $\exists x.A$, if A (or B) is positive then A (resp., B) must have polarity $+R$. A coupled restriction is imposed on sequents (introduced in the next section) of the form $\vdash \Gamma : \Delta \Uparrow^\bullet \Theta$: the multiset Θ may not contain negative formulas except $-L$ formulas. This invariant does not compromise the expressiveness of the logic since one can always switch polarities using any number of unary operations such as $A \otimes 1$.

To capture intuitionistic logic, we impose another restriction that is related to the single-conclusion characterization of intuitionistic logic. We require that in the formula $A \sqcup B$, at least one of A or B must be a left-formula ($+L$ or $-L$). Dually, in $A \boxtimes B$, either A or B must be a right-formula. These connectives are used to model intuitionistic implication. We often write $A \supset B$ for $A^\perp \sqcup B$ with the understanding that A is of right-polarity (so $A^\perp$ is of left-polarity). The dual of this representation of intuitionistic implication is $A \boxtimes B^{\perp}$. The polarity restriction on A allows us to prove, for example, the distributivity of $\boxtimes$ over $\boxplus$ (and of $\sqcup$ over $\sqcap$), which in terms of intuitionistic implication represents the equivalence between $A \supset (B \wedge C)$ and $(A \supset B) \wedge (A \supset C)$. Moreover, this distributivity should hold at the denotational level since it reduces to the distributivity of $\otimes$ over $\oplus$ in linear logic (see the translation of Section 6).

4 Focused Sequent Calculus

The presentation of LUF is based on the focused proof systems LLF [1] and LKU [15]. The introduction rules are kept as uniform as possible while a set of expanded structural rules, which are sensitive to polarity changes, take center stage. Focusing and asynchronous decomposition along the linear +1/-1 axis are represented by $\Downarrow^1 / \Uparrow^1$, along the $+R$/$-L$ axis by $\Downarrow^{\bullet} / \Uparrow^{\bullet}$, and along the $+L$/$-R$ axis by $\Downarrow^{\circ} / \Uparrow^{\circ}$. Sequents of LUF have the form $\vdash \Gamma : \Delta \Uparrow^n \Theta$ or $\vdash \Gamma : \Delta \Downarrow^n A$, where $\Uparrow^n$ denotes $\Uparrow^1$, $\Uparrow^{\bullet}$ or $\Uparrow^{\circ}$ and likewise for $\Downarrow^n$. The multiset Γ is the classical context that admits contraction and weakening and Δ is the linear context. Θ is a unclassified multiset of formulas and A is a single formula under focus. *End sequents* of LUF have the form $\vdash : \Uparrow^1 \Theta$. The choice of this designation is principally due to cut-elimination (see Section A). However, purely classical and intuitionistic end sequents may use $\Uparrow^{\bullet}$ and $\Uparrow^{\circ}$ as well.

Lateral Reactions

$$\frac{\vdash \Gamma : \Uparrow^1 E, \Upsilon}{\vdash \Gamma : \Uparrow^{\circ} E, \Upsilon}\ L\Uparrow^{\circ} \qquad \frac{\vdash \Gamma : \Delta \Uparrow^{\bullet} \Upsilon}{\vdash \Gamma : \Delta \Uparrow^1 \Upsilon}\ L\Uparrow^1 \qquad \frac{\vdash \Gamma : \Delta \Downarrow^1 F}{\vdash \Gamma : \Delta \Downarrow^{\circ} F}\ L\Downarrow^{\circ} \qquad \frac{\vdash \Gamma : \Downarrow^{\bullet} G}{\vdash \Gamma : \Downarrow^1 G}\ L\Downarrow^1$$

Negative Reactions

$$\frac{\vdash \Gamma : \Delta, C \Uparrow^1 \Theta}{\vdash \Gamma : \Delta \Uparrow^1 C, \Theta}\ R_1\Uparrow^1 \qquad \frac{\vdash D, \Gamma : \Delta \Uparrow^{\bullet} \Theta}{\vdash \Gamma : \Delta \Uparrow^{\bullet} D, \Theta}\ R_2\Uparrow^{\bullet} \qquad \frac{\vdash \Gamma : \Delta \Downarrow^1 S}{\vdash \Gamma : \Delta, S \Uparrow^n}\ D_1 \qquad \frac{\vdash T, \Gamma : \Delta \Downarrow^{\circ} T}{\vdash T, \Gamma : \Delta \Uparrow^n}\ D_2$$

Positive Reactions

$$\frac{\vdash \Gamma : \Delta \Uparrow^1 N}{\vdash \Gamma : \Delta \Downarrow^1 N}\ R_1\Downarrow^1 \qquad \frac{\vdash \Gamma : \Uparrow^{\circ} M}{\vdash \Gamma : \Downarrow^{\bullet} M}\ R_2\Downarrow^{\bullet} \qquad \frac{}{\vdash \Gamma : P^{\perp} \Downarrow^n P}\ I_1 \qquad \frac{}{\vdash Q^{\perp}, \Gamma : \Downarrow^n Q}\ I_2$$

E: any formula except a $-R$ non-literal; Υ: all $-L$ and $+L$ formulas; F: not a $+L$ formula; G: $+R$ or $-R$ formula; C: $+1$, $+R$, $-R$ formula or -1 literal; D: $+1$, $+R$, $+L$ formulas and $-L$ literals; S: $+1$, $+R$, or non-literal $-R$ formula; T: $+R$, $+L$ or $+1$ formula; N: -1, $-L$ or $+L$ formula; M: -1, $-L$ or $-R$ formula; P, Q: positive literals.

Fig. 2. LUF Structural Rules

The structural rules are found in Figure 2 while the introduction rules will be given in Figures 3, 4 and 5. The intricate polarity restrictions for the structural rules are balanced by uniform introduction rules. It is the structural rules at the *borders* of the focusing phases that are sensitive to polarity information. They are called "reactions" because they react to changes in polarity. Without the restrictions on these rules, the system would collapse to a verbose version of the unfocused sequent calculus for classical logic. The restrictions are based on the different responsibilities for each mode of sequent. For example, the $\Uparrow^\circ$ mode is responsible for the asynchronous decomposition of $-R$ formulas. Thus when the formula E to the right of $\Uparrow^\circ$ is not a $-R$ formula, or is a mere literal, the $L\Uparrow^\circ$ reaction rule is invoked to transition to another mode.

Note that it is no longer the case that focusing can continue as long as the subformula stays "positive:" see $R_1 \Downarrow^1$, for example. This refinement of focusing allows it to define a wider range of synthetic connectives, as we have explained.

LUF is presented as a one-sided sequent calculus. The richness of polarity information replaces the need for two-sided sequents and allows for a more uniform and compact presentation. A traditional, two-sided sequent will have the left-side formulas moved to the right using De Morgan negation. For example, the sequent $a, a \supset b \vdash b$ can appear in LUF as $\vdash a^\perp, a \boxtimes \top :\Uparrow^\circ b$. One should consider the fact that LUF polarities can mix more freely when judging its cosmetic appearance. Using two-sided sequents for LUF would mean, for example, that one must speak of "left formulas on the right-hand side." A LUF sequent such as $\vdash a^\perp, a \boxtimes \top : r \Downarrow^\circ a \boxtimes \top$, with a non-empty Δ, represents the conclusion of a traditional "left-introduction rule", in this case $\supset L$.

The rules $R_1 \Uparrow$ and $R_2 \Uparrow$ *permeate* formulas into either the linear or classical bin. For example, in the sequent $(\vdash :\Uparrow^\circ A \vee^+ B)$, $R_1 \Uparrow$ applies, which gives us the "disjunction property" at the intuitionistic level. In the sequent $(\vdash :\Uparrow^\bullet A \vee^+ A^\perp)$, $R_2 \Uparrow$ applies, which gives us the law of excluded middle at the classical level. Rules D_1 and D_2 are *decision* rules as they select formulas for focus. D_2, which embodies an explicit contraction, can only select a positive formula. The rules $R_1 \Downarrow^1$ and $R_2 \Downarrow^\bullet$ are also called *release* rules since they terminate focus. The initial rule I_1 can be seen as the "missing case" for $R_1 \Downarrow^1$ and likewise for I_2 in relation to $R_2 \Downarrow^\bullet$. The lateral rules allow one-directional transition of focusing modes. The directions of these transitions are dictated by when structural rules are required. Certain sequences of rules are deterministic. For example, when D_1 selects a $-R$ formula for focus (the only case where a negative formula can be selected for focus), it will immediately trigger a lateral $L \Downarrow^1$ followed by a $R_2 \Downarrow^\bullet$. An important invariant is that the rule $R_2 \Uparrow$ applies to D if and only if $R_2 \Downarrow$ or I_2 applies to $D^\perp$. The polarity conditions of these rules are critical for preserving cut elimination. This invariant refines a similar requirement in our previous system LKU [15].

The number of introduction rules (Figures 3, 4, 5) is much less than in LU. There is only one introduction rule for each LUF connective (taking into account the efficient presentation of additive disjunction). The introduction rules that are most sensitive to polarity information are the rules for $\boxtimes$ and $\sqcup$. In $A \sqcup B$, we will assume, without loss of generality, that A is a left-polarity formula ($+L$ or $-L$). Similarly, in $A \boxtimes B$, we assume that A is a right-formula. Because focusing behaves in an asymmetric way here, these introduction rules incorporate structural rules. The intuitionistic mode of focus can be kept only on B, the head of the implication. For A, the $\Uparrow^\circ / \Downarrow^\circ$ mode must be terminated (see Section 6 for one explanation of this special property).

As in classical logic, some additive and multiplicative versions of disjunction and conjunction are provability-equivalent in LUF. In particular, $\boxplus$ is equivalent to $\vee^-$ and

$$\frac{\vdash \Gamma : \Delta \Uparrow^1 \Theta}{\vdash \Gamma : \Delta \Uparrow^1 \bot, \Theta}\ \bot \qquad \frac{}{\vdash \Gamma : \Delta \Uparrow^1 \top, \Theta}\ \top \qquad \frac{}{\vdash \Gamma : \Downarrow^1 1}\ 1$$

$$\frac{\vdash \Gamma : \Delta \Uparrow^\bullet \Theta}{\vdash \Gamma : \Delta \Uparrow^\bullet \bot_l, \Theta}\ \bot_l \qquad \frac{}{\vdash \Gamma : \Delta \Uparrow^\bullet \top_l, \Theta}\ \top_l \qquad \frac{}{\vdash \Gamma : \Downarrow^\bullet 1_r}\ 1_r \qquad \frac{}{\vdash \Gamma : \Uparrow^\circ \top_r, \Upsilon}\ \top_r$$

Fig. 3. LUF introduction rules for constants. Here, Υ contains only $-L$ or $+L$ formulas.

$$\frac{\vdash \Gamma : \Delta \Uparrow^1 A, B, \Theta}{\vdash \Gamma : \Delta \Uparrow^1 A \parr B, \Theta}\ \parr \qquad \frac{\vdash \Gamma : \Delta \Uparrow^1 A, \Theta \quad \vdash \Gamma : \Delta \Uparrow^1 B, \Theta}{\vdash \Gamma : \Delta \Uparrow^1 A \,\&\, B, \Theta}\ \& \qquad \frac{\vdash \Gamma : \Delta \Uparrow^1 A, \Theta}{\vdash \Gamma : \Delta \Uparrow^1 \Pi x.A, \Theta}\ \Pi$$

$$\frac{\vdash \Gamma : \Delta \Uparrow^\bullet A, B, \Theta}{\vdash \Gamma : \Delta \Uparrow^\bullet A \vee^- B, \Theta}\ \vee^- \qquad \frac{\vdash \Gamma : \Delta \Uparrow^\bullet A, \Theta \quad \vdash \Gamma : \Delta \Uparrow^\bullet B, \Theta}{\vdash \Gamma : \Delta \Uparrow^\bullet A \wedge^- B, \Theta}\ \wedge^- \qquad \frac{\vdash \Gamma : \Delta \Uparrow^\bullet A, \Theta}{\vdash \Gamma : \Delta \Uparrow^\bullet \forall x.A, \Theta}\ \forall$$

$$\frac{\vdash \Gamma : \Uparrow^\circ A, \Upsilon}{\vdash \Gamma : \Uparrow^\circ \forall^r x.A, \Upsilon}\ \forall^r \qquad \frac{\vdash \Gamma : \Uparrow^\circ A, \Upsilon \quad \vdash \Gamma : \Uparrow^\circ B, \Upsilon}{\vdash \Gamma : \Uparrow^\circ A \sqcap B, \Upsilon}\ \sqcap \qquad \frac{\vdash \Gamma : \Uparrow^\circ B, C, \Upsilon}{\vdash \Gamma : \Uparrow^\circ C \sqcup B, \Upsilon}\ \sqcup (\supset R)$$

Fig. 4. LUF introduction rules for the negatives. Here, x is not free in Γ, Δ, Θ; C is $+L$ or $-L$; and Υ contains only $-L$ or $+L$ formulas.

$$\frac{\vdash \Gamma : \Delta_1 \Downarrow^1 A \quad \vdash \Gamma : \Delta_2 \Downarrow^1 B}{\vdash \Gamma : \Delta_1 \Delta_2 \Downarrow^1 A \otimes B}\ \otimes \qquad \frac{\vdash \Gamma : \Delta \Downarrow^1 A_i}{\vdash \Gamma : \Delta \Downarrow^1 A_1 \oplus A_2}\ \oplus \qquad \frac{\vdash \Gamma : \Delta \Downarrow^1 A[t/y]}{\vdash \Gamma : \Delta \Downarrow^1 \Sigma y.A}\ \Sigma$$

$$\frac{\vdash \Gamma : \Downarrow^\bullet A \quad \vdash \Gamma : \Downarrow^\bullet B}{\vdash \Gamma : \Downarrow^\bullet A \wedge^+ B}\ \wedge^+ \qquad \frac{\vdash \Gamma : \Downarrow^\bullet A_i}{\vdash \Gamma : \Downarrow^\bullet A_1 \vee^+ A_2}\ \vee^+ \qquad \frac{\vdash \Gamma : \Downarrow^\bullet A[t/y]}{\vdash \Gamma : \Downarrow^\bullet \exists y.A}\ \exists$$

$$\frac{\vdash \Gamma : \Delta \Downarrow^\circ A[t/y]}{\vdash \Gamma : \Delta \Downarrow^\circ \exists^l y.A}\ \exists^l \qquad \frac{\vdash \Gamma : \Delta \Downarrow^\circ A_i}{\vdash \Gamma : \Delta \Downarrow^\circ A_1 \boxplus A_2}\ \boxplus \qquad \frac{\vdash \Gamma : \Downarrow^1 D \quad \vdash \Gamma : \Delta \Downarrow^\circ B}{\vdash \Gamma : \Delta \Downarrow^\circ D \boxtimes B}\ \boxtimes (\supset L)$$

Fig. 5. LUF introduction rules for the positive connectives. Here, D is either $+R$ or $-R$.

$\sqcap$ is equivalent to $\wedge^+$. The differences they bring are in the constructive content of focused proofs. For example, one may implement a forward-chaining proof strategy by choosing the positive versions of the connectives.

5 Basic Fragments

Focused versions of other logics are found as fragments of LUF. In all cases the *restriction* is only on the forms of formulas and the end-sequent, not on how a proof should be constructed. A proof of, for example, an intuitionistic end sequent will naturally stay within the intuitionistic fragment. In addition to focused versions of classical logic (LKF), intuitionistic logic (LJF) and MALL, it is also possible to identify new logics. We describe some of the fragments below:

MALLF : Restrict to only +1/-1 formulas and use $\vdash :\Uparrow^1 \Theta$ for the end sequent. The only applicable structural rules are $R_1 \Uparrow^1$, $R_1 \Downarrow^1$, D_1, and I_1. The resulting proof system is the MALL subset of Andreoli's proof system [1].

It is also easy to show that *LUF has the full power of linear logic.* Given a MALL formula A, $?A$ can be recovered with $A \boxplus 0$, $(A \otimes 1) \vee^- \bot_l$ or any number of other forms. $!A$ is recovered from the duals of these forms (see the translation of Section 6).

LKF : Restrict to only $+R/-\mathrm{L}$ formulas and to end sequents of the form $\vdash:\Uparrow^\bullet \Theta$ or $\vdash:\Uparrow^1 A_1 \vee^- \ldots \vee^- A_n \vee^- \bot_l$. The only applicable structural rules are $R_2 \Uparrow^\bullet$, $R_2 \Downarrow^\bullet$, D_2, I_2, and the lateral reactions for transition after a decide/release rule. Within the LKF fragment, $\vee^+$ and $\vee^-$ are provably equivalent, as are $\wedge^-$ and $\wedge^+$.

LJF : LJF formulas, as they originally appeared in [13] are mapped into LUF formulas using the two functions $[\cdot]^R$ (right) and $[\cdot]^L$ (left) defined in Figure 6. Thus, LJF formulas only employ connectives of polarity $+L$, $-L$, $+R$ and $-R$. Atoms are restricted to $+R$ and $-R$. It should be noted that the $\wedge^-$ symbol used in the original presentation of LJF is replaced by $\sqcap$ here and is not to be confused with $\wedge^-$ in LKF and LUF.

Intuitionistic negation is represented by $A \supset 0_r$. For minimal logic, replace 0_r with some designated $-R$ or $+R$ atom.

$$
\begin{array}{ll}
[B \wedge^- C]^R = [B]^R \sqcap [C]^R & [B \wedge^+ C]^R = [B]^R \wedge^+ [C]^R \\
[B \supset C]^R = [B]^L \sqcup [C]^R & [B \vee C]^R = [B]^R \vee^+ [C]^R \\
[\forall x.B]^R = \forall^r x.[B]^R & [\exists x.B]^R = \exists x.[B]^R \\
[B \wedge^- C]^L = [B]^L \boxplus [C]^L & [B \wedge^+ C]^L = [B]^L \vee^- [C]^L \\
[B \supset C]^L = [B]^R \boxtimes [C]^L & [B \vee C]^L = [B]^L \wedge^- [C]^L \\
[\forall x.B]^L = \exists^l x.[B]^L & [\exists x.B]^L = \forall x.[B]^L
\end{array}
$$

For atomic A, $[A]^R = A$ and $[A]^L = A^\perp$.

Fig. 6. Left and Right Intuitionistic Formulas in LUF.

Formulas in the range of $[\cdot]^R$ are called *essentially right* and those in the range of $[\cdot]^L$ are *essentially left.* End-sequents of LJF have the form $\vdash:\Uparrow^\circ \Gamma, A$ where Γ consists of essentially left formulas and A is also an essentially right formula[3].

The rules covering intuitionistic implication in LUF use polarity information when splitting the context Δ as a result of applying the $\boxtimes$ rule. In contrast to formulations of intuitionistic logic in linear logic and in LU (and LKU), there is no loss of "full completeness" with respect to intuitionistic implication or to focusing. Specifically, if implication (on the left) is represented with a multiplicative conjunction, then splitting the context may leave two right-formulas in the same sequent. But since one of the immediate subformulas of the conjunct is always a right-formula, for which the rule $L \Downarrow^1$ will enforce an empty linear context, Δ must be moved completely to the subproof containing the (possibly) non-right formula. In linear logic, the ! operator conveys this information. However, to preserve *focused proofs* this operator must be strategically removed *along with the polarity information it carries.* A linear logic translation of LJF

[3] If the end sequent was of the form $\vdash:\Uparrow^1 \Gamma, A$ and A is a $-R$ formula, then the proof would unnecessarily delay the asynchronous decomposition of A.

cannot preserve valid proofs when the $\top$ rule is used, and it cannot preserve partial proofs. The novelty of LUF is in decomposing the ! into two polarities, $+R$ and $-R$. *A LUF formula hence carries more information than a linear logic formula.* A partial LJF proof corresponds one-to-to one with a partial LUF proof.

nLJF : Restrict LJF to only $-R$ and $+L$ connectives: *i.e.*, only to the left-side of Figure 6 and only with -R literals. This fragment of intuitionistic logic is traditionally referred to as the "negative" fragment. The applicable structural rules of nLJF are I_1, D_2, $R_1 \Uparrow^1$ (on -R literals), $R_2 \Uparrow^\bullet$ (on $+L$ formulas) and $R_2 \Downarrow^\bullet$. nLJF fits completely within one axis of polarization, using a single pair of arrows, $\Uparrow^\circ / \Downarrow^\circ$, except when vacuous laterals are needed to invoke the appropriate structural rules.

Except for the trivial use of the lateral reactions for purely bureaucratic reasons, there is in fact no need for lateral reaction rules shifting one focusing or decomposition mode to another in any of the above fragments. In LJF, the forms of sequents involved are $\Downarrow^\bullet$ and $\Uparrow^\circ$ on essentially right formulas and $\Uparrow^\bullet$ and $\Downarrow^\circ$ on left formulas. The only lateral transition is from $\Downarrow^\circ$ to $\Downarrow^\bullet$ in the $\boxtimes$-rule ($\supset$-Left in traditional presentations of intuitionistic logic), and the corresponding transition from $\Uparrow^\circ$ to $\Uparrow^\bullet$ for $\sqcup$. However, the restricted form of intuitionistic formulas makes other transitions unnecessary. The basic fragments above can each be independently represented using a single pair of $\Uparrow$ / $\Downarrow$, and thus do not represent the full potential of LUF. Other fragments of LUF can be identified, such as the following.

ACMALL: Additive Classical Logic with MALL : This fragment is based on all of MALL plus the connectives $\sqcap$, $\boxplus$, $\forall^r$ and $\exists^l$. All formulas can mix freely among these polarities. However, for purely classical reasoning one must use the form

$$\vdash:\Uparrow^1 A_1 \boxplus \ldots \boxplus A_n \boxplus 0_l$$

for classical end-sequents. The modes $\Uparrow^1 / \Downarrow^1$ and $\Uparrow^\circ / \Downarrow^\circ$ are both used in ACMALL.

Polarized Intuitionistic Logic : The most significant new logic that we have derived as a fragment of LUF is a subsystem that leaves out the linear $+1/-1$ axis. Intuitionistic logic is combined with classical logic. In addition to its focused form as a fragment of LUF, this logic has an independent Kripke-style semantics and unique tableau-style proof system. It has a decidable propositional fragment but contains two levels of consistency based on $\perp$ and 0. This system is described in a separate paper [16].

6 Translation to Linear Logic

We provide a translation of LUF into linear logic which will allow LUF to inherit some of the semantics of linear logic. The translation is adequate at the level of formulas (in terms of their provability) but, as explained previously, cannot be expected to preserve all proofs and proof fragments.

To assign polarity to atoms, we admit into linear logic, as was done in LU and polarized linear logic [11], atoms that are naturally subject to structural rules: i.e., $A \equiv ?A$ or $A \equiv !A$. However, unlike in LU, we do not fix the $+/-$ polarity of these atoms.

Conceptually, one can describe the translation of LUF as being based on synthetic connectives derived from the following forms:

$A \wedge^+ B$	$!(!A \otimes !B)$	$A \boxplus B$	$?(?A \oplus ?B)$
$A \vee^+ B$	$!(!A \oplus !B)$	$A \sqcap B$	$!(!A \,\&\, !B)$
$A \wedge^- B$	$?(?A \,\&\, ?B)$	$A \sqcup B$	$!(?A \parr !B)$
$A \vee^- B$	$?(?A \parr ?B)$	$A \boxtimes B$	$?(!A \otimes ?B)$

Of course, to see that these forms are indeed valid synthetics would require a refined notion of focusing, as we have explained in Section 2.

This translation (if made recursive), already allows LUF to inherit the phase space - topolinear space semantics of formulas [5] (read ! as *interior* and ? as *closure*). However, we are more interested in how far *focused proofs* can be preserved. Although these forms clearly do not allow focusing in linear logic, there are equivalences such as $!(!A \otimes !B) \equiv !A \otimes !B$. That is, the ! can be dropped on subformulas of the same polarity. These equivalences apply to what we refer to as $+R$ and $-L$ formulas, and are valid for both multiplicatives and additives (including quantifiers). The $+R$/$-L$ polarities are enough to account for classical logic (LC, LK$^\eta$ and LKF). A representation of intuitionistic logic using only these polarities is not satisfactory with respect to the most important intuitionistic connective: $\supset$. Focusing is indifferent to the additive/multiplicative distinction in both linear and classical logic. With the polarities $+L$ and $-R$, a distinction appears, which hints at a generalization of intuitionistic implication. With the additives we have that $?(A \oplus B) \equiv ?(?A \oplus ?B)$. The internal ?s can be dropped so focusing can continue, even when switching from an intuitionistic context to a classical (or linear) one. There is no equivalence, however, between $!(!A \parr !B)$ and $!(A \parr B)$. With the multiplicatives, we only have the equivalence $!(?A \parr !B) \equiv !(?A \parr B)$ and its dual in terms of $\otimes$. This form is adequate for intuitionistic implication: $A \supset B$ can be translated as $!(?A^\perp \parr B)$. The external ! cannot be dropped if we wish to use these formulas in a unified setting where intuitionistic, classical and linear formulas can exist in the same sequent.

The formal translation is based on the possible polarities for each formula and is given by Table 1. To minimize the number of cases that need to presented, we note that the translation of the two subformulas of binary connectives are independent of each other. Thus we shall only display cases where the polarities of the two subformulas are different. For the cases of the classical/intuitionistic connectives, we also do not show the cases that can be inferred by duality.

The linear connectives (such as $\parr$) in LUF do *not* always translate to themselves in linear logic. When the linear connectives join $-R$ and $+L$ formulas, we "impart" the appropriate exponential operator onto them.

All right-polarity formulas A translate into forms A' such that $A' \equiv !A'$ regardless of whether A is positive or negative. Likewise, for all left-polarity formulas B, $B' \equiv ?B'$. An important difference between LUF and previous studies of polarization is the decoupling of ! and ? from their status as positive and negative operators respectively. A formula of the form $!(A \,\&\, B)$ is considered negative: its synthetic introduction is *invertible* in an intuitionistic context.

The LUF end-sequent $\vdash:\Uparrow^1 A_1, \ldots, A_n$ is translated into LLF as $\vdash:\Uparrow (A_1 \parr \ldots \parr A_n \parr \perp)'$. The "classical" end-sequent $\vdash:\Uparrow^\bullet A_1, \ldots, A_n$ is translated as $\vdash:\Uparrow (A_1 \vee^- \ldots \vee^- A_n \vee^- \perp_l)'$. The "intuitionistic" end-sequent $\vdash:\Uparrow^\circ \Gamma, A$ where A is a right-formula, is translated in the same way as general LUF sequents *with the* ! *in font of* A' *removed if A is a -R formula.* Left formulas in Γ translate to forms that are equivalent to

A	B	$(A \parr B)'$	$(A\&B)'$	$(A \otimes B)'$	$(A \oplus B)'$	$(\Pi x.A)'$	$(\Sigma x.B)'$
+1	–1	$A' \parr B'$	$A'\&B'$	$A' \otimes B'$	$A' \oplus B'$	$\Pi x.A'$	$\Sigma x.B'$
$+L$	$-R$	$?A' \parr !B'$	$?A'\&!B'$	$?A' \otimes !B'$	$?A' \oplus !B'$	$\Pi x.?A'$	$\Sigma x.!B'$
$+R$	$-L$	$A' \parr B'$	$A'\&B'$	$A' \otimes B'$	$A' \oplus B'$	$\Pi x.A'$	$\Sigma x.B'$

There are some cases for the quantifiers that are missing from the above table: for $+L$ formula A and $-R$ formula B, $(\Pi x.B)' = \Pi x.!B$ and $(\Sigma x.A)' = \Sigma x.?A'$. The other cases can be inferred by duality.

A	B	$(A \wedge^+ B)'$	$(A \boxtimes B)'$	$(A \vee^+ B)'$	$(A \boxplus B)'$	$(\exists x.A)'$	$(\exists^l x.B)'$
+1	–1	N/A	N/A	N/A	$A' \oplus B'$	N/A	$\Sigma x.B'$
$+L$	$-R$	N/A	$A' \otimes !B'$	N/A	$A' \oplus !B'$	N/A	$\Sigma x.!B'$
$+R$	$-L$	$A' \otimes !B'$	$A' \otimes B'$	$A' \oplus !B'$	$A' \oplus B'$	$\Sigma x.A'$	$\Sigma x.B'$
$-L$	–1	$!A' \otimes !B'$	N/A	$!A' \otimes !B'$	$A' \oplus B'$	$\Sigma x.!A'$	$\Sigma x.B'$
–1	$-R$	$!A' \otimes !B'$	$A' \otimes !B'$	$!A' \oplus !B'$	$A' \oplus !B'$	$\Sigma x.!A'$	$\Sigma x.!B'$
$+L$	$+L$	N/A	N/A	N/A	$A' \oplus B'$	N/A	$\Sigma x.B'$
$+L$	$-L$	N/A	N/A	N/A	$A' \oplus B'$	N/A	$\Sigma x.B'$
$-R$	$+R$	$!A' \otimes B'$	$!A' \otimes B'$	$!A' \oplus B'$	$!A' \oplus B'$	$\Sigma x.!A'$	$\Sigma x.B'$

The translation of atoms and the constants 1_r, 0_r, 0_l, $\top_l$, $\top_r$ and $\bot_l$ are invariant.

Table 1. Translation of LUF to linear logic.

?-formulas, so the invertible promotion is applicable. This will allow an asynchronous right formula to be eagerly decomposed.

The correctness of this translation of LUF can be shown by observing the correspondence between LUF proofs and LLF proofs of its image. The translation preserves provability. Expectedly however, one LUF proof may correspond to several LLF proofs if the $\top$ rule is involved. It also certainly does not preserve partial proofs as discussed in Section 5. However, as observed in [7], proof fragments using the $\top$ rule can have the same, vacuous denotational interpretation in linear logic (as the empty clique of the empty coherent space). It is questionable if a semantics of proofs should disregard the $\top$ rule entirely, since it has computational content in the form of the *abort* operator. Nevertheless, the translation allows LUF to inherit at least one form of semantics of linear logic in so far as complete proofs are concerned.

7 Cut Elimination, Contraction and the Degree of Polarization

The origins of the "classical" polarities $+R$/$-L$ can be attributed to Girard's LC and LU systems. The polarities +1 and -1 originate from focusing in linear logic as defined by Andreoli. What is new in LUF is the introduction of the polarities $+L$/$-R$ and the amalgamation of all six polarities into a single system. The need for the new axis of polarization can be explained in terms of cut-elimination, especially when some *but not all* formulas are subject to contraction. Consider the reduction of cut above a contraction:

$$\dfrac{\vdash A, \Delta \quad \dfrac{\vdash A^\perp, A^\perp, \Gamma}{\vdash A^\perp, \Gamma}\, C}{\vdash \Delta\Gamma}\, cut \qquad \longmapsto \qquad \dfrac{\dfrac{\vdash A, \Delta \quad \dfrac{\vdash A, \Delta \quad \vdash A^\perp, A^\perp, \Gamma}{\vdash A^\perp, \Delta\Gamma}\, cut}{\vdash \Delta\Delta\Gamma}\, cut}{\vdash \Delta\Gamma}\, C^*$$

If the context Δ contain linear formulas not subject to contraction, then this reduction cannot be made. Intuitionistic logic has been described as "classical on the left, linear on the right" and in many ways LUF is an extension of intuitionistic principles. We seek to extend the already-hybrid characteristics of intuitionistic logic. To ensure the admissibility of cut in this setting, we can consider three approaches.

The first approach is the traditional intuitionistic restriction to a "single conclusion" (i.e., the cut formula is the only linear formula). This approach is enough if one is only interested in embedding intuitionistic logic. But it is not adequate if we wish to aggressively mix intuitionistic deduction with linear deduction. A second approach involves using the ! of linear logic. The !, however, is not compatible with focusing; indeed it obscures the synchronous/asynchronous duality. In the terminology of Girard, this problem can also be described as the loss of denotational associativity. The third approach is to replace ! (and thus ?) with polarity information. LU was the first system to make such a use of polarities. However, in LU the "negative" intuitionistic formulas, along with $+1$ and -1 formulas, were all classified as having "neutral" polarity; that is, linear. Although LU's translation tables also suggest that mixing connectives from different logics is possible, this aspect of LU was never fully explored. LU includes classical, intuitionistic and linear logics as fragments but its ability to mix connectives in a single formula (i.e., identify new fragments) is, in fact, limited. Its admissible cuts do not include, for example, the following case:

$$\dfrac{;\Gamma \vdash ; A, \Delta \quad ; A, \Gamma' \vdash ; \Delta'}{;\Gamma\Gamma' \vdash ; \Delta\Delta'}\, cut$$

where A is a "negative" intuitionistic formula and Δ is a non-empty context of formulas that are not subject to contraction. The problem is that the polarity of A is not distinguished from that of linear formulas and consequently the introductions rules for such formulas may take place in the presence of other linear formulas. In contrast, a cut between $!A$ and $?A^\perp$ is admissible in *any* context in linear logic: the permutation of such a cut above a contraction can be delayed until the promotion rule is applied. The polarization scheme of LU is not enough to completely replace the role of the exponential operators or the single-conclusion restriction. More polarities are needed.

By considering where structural rules are needed in a *focused* proof, a refined polarization scheme emerges. The "right" polarities of LUF, $+R$ and $-R$, are equivalent (provability-wise) to !-formulas in linear logic, but unlike Andreoli's system, they are not both considered "positive". Focusing on $+R$ formulas requires an empty linear context. The $+L$ polarity enables the focusing of left-introduction rules in intuitionistic logic, which must take place in the presence of a *non-empty* linear context (consisting of a single right-polarity formula). The $+L$ polarity must, therefore, be distinguished from $+R$. The focusing of a left-side intuitionistic formula should be *preceded by a contraction.* Thus the $+L$ polarity should also be distinguished from +1, which indicates positive MALL formulas. The contraction at the border of the focusing phase entails that inference rules for the dual polarity -R require an empty linear context. Thus the -R polarity should likewise be distinguished from -1. The fine-grained sensitivity to cut-elimination found in LUF is possible because of the additional polarities and modes of focusing. Cut elimination will fail if, for example, we were to allow arbitrary transitions from the $\Uparrow^{\bullet}$ mode to the $\Uparrow^1$ mode (i.e., allow the inverse of $L\Uparrow^1$).

The consequence of the enriched polarization scheme is a general form of admissible cut that is in fact *independent of polarity restrictions.* We refer to it as the *end cut* (also referred to as "strong cut" elsewhere) as it applies to LUF end-sequents:

$$\frac{\vdash:\Uparrow^1 A, \Theta \quad \vdash:\Uparrow^1 A^{\perp}, \Theta'}{\vdash:\Uparrow^1 \Theta\Theta'}\ Cut$$

During cut-reduction *starting with the end cut,* the following cuts are also needed:

$$\frac{\vdash \Gamma : \Delta, A \Uparrow^n \Theta \quad \vdash \Gamma' : \Delta' \Uparrow^m A^{\perp}, \Theta'}{\vdash \Gamma\Gamma' : \Delta\Delta' \Uparrow^1 \Theta\Theta'}\ cut_1 \qquad \frac{\vdash \Gamma, A : \Delta \Uparrow^n \Theta \quad \vdash \Gamma' :\Uparrow^m A^{\perp}, \Upsilon}{\vdash \Gamma\Gamma' : \Delta \Uparrow^n \Theta\Upsilon}\ cut_2$$

The multiset Υ may contain only left-formulas. In the cut_1 rule, $\Uparrow^n$ cannot be $\Uparrow^{\circ}$ and in cut_2, $\Uparrow^m$ cannot be $\Uparrow^{\circ}$. Rules similar to cut_1 and cut_2 are found in LU and LKU. They would be enough if one only considered classical, linear and intuitionistic logics as *independent* fragments. In the expanded setting of LUF, the usual consequences and applications of cut-elimination are only possible in general with the stronger Cut rule. These consequences include, for example, the disjunction property in terms of $\vee^+$ and $\oplus$ and the existence property in terms of $\exists$ and Σ.

The detailed proof of cut-elimination, which has been carried out but is omitted here[4], also takes advantage of the structure of focused proofs. Positive and negative introduction rules are generalized to allow us to concentrate on cut reduction only where it matters the most: at the borders marking polarity transitions.

Theorem 1. *The Cut, cut_1 and cut_2 rules are admissible in LUF.*

Notably, because every fragment of LUF is self-restricting (cut-free proofs stay naturally within each fragment), *the cut-elimination proof of the unified logic is also a cut-elimination proof for each of its fragments.* Furthermore, as in LU, it is possible for proofs in different fragments to interact using cut-formulas that the fragments have in common. However, the main aim of LUF is to allow connectives of different polarities to mix in the same formula. Initial elimination can also be proved. This property tests if the connectives of our logic are *"small enough".*

Theorem 2. *For all formulas A, the sequent $\vdash:\Uparrow^1 A, A^{\perp}$ is provable in LUF.*

8 Conclusion

We have developed a unified proof-theoretic system using a refined analysis of polarization. Two of the dimensions of polarization stems from that of Girard's LC, and

[4] A version of this paper with an appendix on the cut-elimination proof is available at **www.cs.hofstra.edu/∼cscccl/lufpaper.pdf**.

from Andreoli's notion of focusing. We have added a generalization of the left-right distinction of intuitionistic logic as a further dimension. By refining the polarity information attached to connectives, and the focused proofs that use this information, we can embed intuitionistic, classical and multiplicative-additive linear logic as fragments. Full-adequacy is achieved in that, for example, an embedded intuitionistic sequent is only subject to valid intuitionistic inference rules. Furthermore, the formulas of these logics can mix with few restrictions.

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A The Cut Elimination Proof

We provide in this detachable appendix further details of the cut-elimination proof (as well as initial elimination). The proof is in fact for a version of LUF with slightly more general versions of the $\boxtimes$ and $\vee^-$ connectives, which does not have the restriction that one of the immediate subformulas must be of right (left) polarity. The arguments can be easily adjusted for LUF as presented.

Because there are few restrictions on how formulas can compose in LUF, general cut-elimination may at first be hard to believe. In particular, a positive linear (+1) formula can enter the classical context (Γ) as a result of $R_2 \Uparrow^\bullet$. A pair of sequents of the forms $\vdash A, \Gamma : \Delta \Uparrow^\bullet$ and $\vdash \Gamma' : B, \Delta' \Uparrow^\bullet A^\perp$ cannot be joined with an admissible cut. If the cut is permuted above a contraction on A, two copies of B would form when we try to stack the cuts. The most important subtlety of cut-elimination in a mixed classical-linear context, without the explicit ! and ?, is to show that this situation cannot occur. Cut elimination will in fact fail if, for example, we were to allow arbitrary transitions from $\Uparrow^\bullet$ to $\Uparrow^1$ (i.e., allow the inverse of $L \Uparrow^1$). The form of the LUF *end sequent,* $\vdash:\Uparrow^1\Theta$, is an important designation. It suffices to prove cut-elimination for end sequents:

End cut:

$$\frac{\vdash:\Uparrow^1 A, \Theta \quad \vdash:\Uparrow^1 A^\perp, \Theta'}{\vdash:\Uparrow^1 \Theta\Theta'}\ Cut$$

We will show that during the permutation of cuts *starting with the end cut,* only the following valid forms of cuts will be needed:

Intermediate cuts:

$$\frac{\vdash \Gamma : \Delta, A \Uparrow^n \Theta \quad \vdash \Gamma' : \Delta' \Uparrow^m A^\perp, \Theta'}{\vdash \Gamma\Gamma' : \Delta\Delta' \Uparrow^1 \Theta\Theta'}\ cut_1 \qquad \frac{\vdash \Gamma, A : \Delta \Uparrow^n \Theta \quad \vdash \Gamma' :\Uparrow^m A^\perp, \Upsilon}{\vdash \Gamma\Gamma' : \Delta \Uparrow^n \Theta\Upsilon}\ cut_2$$

$$\frac{\vdash B, \Gamma : \Delta \Uparrow^n \Theta \quad \vdash \Gamma' : \Delta', B^\perp \Uparrow^m \Theta'}{\vdash \Gamma\Gamma' : \Delta\Delta' \Uparrow^1 \Theta\Theta'}\ cut_\circ$$

In the $cut_\circ$ rule, B is a +L formula: this special form is needed because a -R formula is not always immediately decomposed. $cut_\circ$ will always permute to a cut_2. Υ may contain only left formulas: it is needed because a $L \Uparrow^1$ lateral may not be possible due to the polarity of $A^\perp$. In the cut_1 rule, $\Uparrow^n$ cannot be $\Uparrow^\circ$ and in cut_2, $\Uparrow^m$ cannot be $\Uparrow^\circ$. Also for cut_1, if both $\Uparrow^n$ and $\Uparrow^m$ are $\Uparrow^\bullet$, then the cut-free provability of the conclusion will immediately imply that $\vdash \Gamma\Gamma' : \Delta\Delta' \Uparrow^\bullet \Theta\Theta'$ is also provable via a $L\Uparrow^1$ transition[5]. We thus also include the following form for convenience:

$$\frac{\vdash \Gamma : \Delta, A \Uparrow^\bullet \Theta \quad \vdash \Gamma' : \Delta' \Uparrow^\bullet A^\perp, \Theta'}{\vdash \Gamma\Gamma' : \Delta\Delta' \Uparrow^\bullet \Theta\Theta'}\ cut_1$$

[5] Another alternative is to use $\Uparrow^\circ$, and allow transitions from any non-valid intuitionistic forms via $L\Uparrow^\circ$.

These cuts are also assumed to observe other invariants on LUF sequents. For example,

$$\frac{\vdash \Gamma, A \otimes B : \Delta \Uparrow^{\bullet} \Theta \quad \vdash \Gamma' : \Uparrow^{\bullet} A^{\perp} \parr B^{\perp}}{\vdash \Gamma\Gamma' : \Delta \Uparrow^{\bullet} \Theta}\ cut_2$$

is not valid because a -1 formula cannot be on the right side of $\Uparrow^{\bullet}$. Only with the end cut, which must be stated in terms of $\Uparrow^1$ and $\Downarrow^1$, is there no restriction on the polarity of formulas. Significantly, one of the cut-formulas of the end sequent will be sent to the linear context (Δ) via $R_1 \Uparrow^1$. That is to say that the end-cut will first permute to a cut_1.

Generalizing the Phases

It is known that during cut-elimination for focused systems, cuts can be permuted across entire synchronous and asynchronous phases. For LUF, however, we also need to take into account the *lateral borders.* The style of the cut-elimination proof here is based on that of [14]. We generalize the synchronous and asynchronous phases and first show that the introduction rule does not cause any problems with cut-elimination.

If Φ is a multiset of formulas $\{a_1, \ldots, a_n\}$, we write $\Phi^{\perp}$ to represent the multiset $\{a_1^{\perp}, \ldots, a_n^{\perp}\}$. As in the notation for sequents, we write $\Phi\Phi'$ to denote the multiset union of Φ and Φ'.

Definition 1. *Define the following relations between formulas and pairs of multisets of formulas:*

- $\circ\ A \downarrow^{\bullet} (\{A\}, \{\})$ *if* A *is a +R literal, or negative formula.*
- $\circ\ 1_r \downarrow^{\bullet} (\{\}, \{\})$.
- $\circ\ (A \wedge^{+} B) \downarrow^{\bullet} (\Psi\Psi', \{\})$ *if* $A \downarrow^{\bullet} (\Psi, \{\})$ *and* $B \downarrow^{\bullet} (\Psi', \{\})$.
- $\circ\ (A \vee^{+} B) \downarrow^{\bullet} (\Psi, \{\})$ *if* $A \downarrow^{\bullet} (\Psi, \{\})$.
- $\circ\ (A \vee^{+} B) \downarrow^{\bullet} (\Psi', \{\})$ *if* $B \downarrow^{\bullet} (\Psi', \{\})$.

- $\circ\ A \downarrow^{1} (\{\}, \{A\})$ *if* A *is a +1 literal, +L, -1, or -L formula.*
- $\circ\ A \downarrow^{1} (\{A\}, \{\})$ *if* A *is a -R formula.*
- $\circ\ A \downarrow^{1} (\{A\}, \{\})$ *if* A *is a +R formula (lateral border).*
- $\circ\ 1 \downarrow^{1} (\{\}, \{\})$.
- $\circ\ (A \otimes B) \downarrow^{1} (\Psi_A^2\Psi_B^2, \Psi_A^1\Psi_B^1)$ *if* $A \downarrow^{1} (\Psi_A^2, \Psi_A^1)$ *and* $B \downarrow^{1} (\Psi_B^2, \Psi_B^1)$.
- $\circ\ (A \oplus B) \downarrow^{1} (\Psi_A^2, \Psi_A^1)$ *if* $A \downarrow^{1} (\Psi_A^2, \Psi_A^1)$*;*
- $\circ\ (A \oplus B) \downarrow^{1} (\Psi_B^2, \Psi_B^1)$ *if* $B \downarrow^{1} (\Psi_B^2, \Psi_B^1)$*;*

- $\circ\ A \downarrow^{\circ} (\Psi_A^2, \Psi_A^1)$ *if* A *is not a +L formula and* $A \downarrow^{1} (\Psi_A^2, \Psi_A^1)$ *(lateral transition).*
- $\circ\ (A \boxplus B) \downarrow^{\circ} (\Psi_A^2, \Psi_A^1)$ *if* $A \downarrow^{\circ} (\Psi_A^2, \Psi_A^1)$*;*
- $\circ\ (A \boxplus B) \downarrow^{\circ} (\Psi_B^2, \Psi_B^1)$ *if* $B \downarrow^{\circ} (\Psi_B^2, \Psi_B^1)$.
- $\circ\ (D \boxtimes B) \downarrow^{\circ} (\Psi_D^2\Psi_B^2, \Psi_D^1\Psi_B^1)$ *if* $D \downarrow^{1} (\Psi_D^2, \Psi_D^1)$ *and: if* D *is a right formula then* $B \downarrow^{\circ} (\Psi_B^2, \Psi_B^1)$*, else* $B \downarrow^{1} (\Psi_B^2, \Psi_B^1)$[6].

The asynchronous relations are defined by duality:

[6] in case D is a +R/-R formula, Ψ_D^1 will be empty, a fact ensured by other cases.

$\circ$ $A \uparrow^{\bullet} (\Psi^{\perp}, \{\})$ *if* $A^{\perp} \downarrow^{\bullet} (\Psi, \{\})$
$\circ$ $A \uparrow^{1} (\Psi^{\perp}, \Psi'^{\perp})$ *if* $A^{\perp} \downarrow^{1} (\Psi, \Psi')$
$\circ$ $A \uparrow^{\circ} (\Psi^{\perp}, \Psi'^{\perp})$ *if* $A^{\perp} \downarrow^{\circ} (\Psi, \Psi')$

(The quantifiers $\forall, \exists$ can be treated similarly as long as we take measures to respect the names of bound variables.)

Given a formula A, $A \downarrow^{\circ} (\Psi^2, \Psi^1)$ represents a possible, synchronous decomposition of A *up to* the lateral border defined by $L \Downarrow^1$. Formulas in Ψ^2 require a confirmation of an empty linear context before proceeding, in the form a $L \Downarrow^1$, $R_2 \Downarrow^{\bullet}$ or I_2 rule. Formulas in Ψ^1 are conclusions of an $R_1 \Downarrow^1$ or I_1 rule. Dually, $A \uparrow^{\circ} (\Phi^2, \Phi^1)$ represents a possible asynchronous decomposition. It is here that care must be taken, since we can not completely decompose a single formula because of the requirement at the lateral border defined by $L \Uparrow^1$. Formulas in Φ^2 will be sent via $R_2 \Uparrow^{\bullet}$ to the classical context *after* a transition to $\Uparrow^{\bullet}$, and Φ^1 formulas will be sent to the linear context. The border between the $\Uparrow^{\circ}/ \Downarrow^{\circ}$ and $\Uparrow^1/ \Downarrow^1$ phases are less carefully "guarded" because the requirements for cut-elimination actually *relaxes* in the later modes.

To extend the decomposition relations past the lateral borders, we extend the $\uparrow^n$ / $\downarrow^n$ relations to multisets:

$\circ$ $\{a_1, \ldots, a_n\} \uparrow^n (\phi_1^2 \ldots \phi_n^2, \phi_1^1 \ldots \phi_n^1)$ if $a_i \uparrow^n (\phi_i^2, \phi_i^1)$ for each $1 \leq i \leq n$.
$\circ$ $\Theta \downarrow^n (\Psi^2, \Psi^1)$ if $\Theta^{\perp} \uparrow (\Psi^{2\perp}, \Psi^{1\perp})$

A *complete asynchronous decomposition* of A in mode $\Uparrow^n$ is described by $A \uparrow^n (\Upsilon, \Phi^1)$ and $\Upsilon \uparrow^{\bullet} (\Phi^2, \{\})$ (and dually for $\downarrow^n$). We use the symbol $\nearrow_n$ and $\searrow^n$ to represent complete decompositions with respect to $\Uparrow^n$ and $\Downarrow^n$. These relations cross the "lateral borders." It holds that $A \nearrow_n (\Phi_2, \Phi_1)$ if and only if $A^{\perp} \searrow^n (\Phi_2^{\perp}, \Phi_1^{\perp})$.

These relations characterize the structure of synchronous and asynchronous phases in a proof. For purely managerial reasons we shall admit a vacuous lateral transition of the form

$$\frac{\vdash \Gamma : \Delta \Uparrow^{\bullet}}{\vdash \Gamma : \Delta \Uparrow^{n}} \; L\Uparrow$$

The sole purpose of this rule is so we can assume a uniform type of $\Uparrow$ after each complete asynchronous decomposition phase (i.e., before the next decide rule).

It is also possible to give a separate definition for the $\uparrow^n$ relations, then prove as a lemma the duality of $\downarrow^n$ / $\uparrow^n$. In any case one can prove the following:

Lemma 1. *Given a formula* R*, let* $(\Upsilon_1, \Phi_1^1), \ldots, (\Upsilon_m, \Phi_m^1)$ *be pairs of multisets such that* $R \uparrow^n (\Upsilon_1, \Phi_1^1), \ldots, R \uparrow^n (\Upsilon_m, \Phi_m^1)$ *and if* $R \uparrow^n \Phi$ *then* $\Phi = (\Upsilon_i, \Phi_i^1)$ *for some unique* $1 \leq i \leq m$[7].

Every cut-free proof of $\vdash \Gamma : \Delta \Uparrow^n R, \Theta$ *has the form*

$$\dfrac{\dfrac{\vdash \Gamma : \Delta\Phi_1^1 \Uparrow^n \Upsilon_1 \Theta}{\vdots} \quad \cdots \quad \dfrac{\vdash \Gamma : \Delta\Phi_m^1 \Uparrow^n \Upsilon_m \Theta}{\vdots}}{\vdash \Gamma : \Delta \Uparrow^n R, \Theta}$$

[7] in the following we shall simple assume that such as list is *exhaustive.*

Furthermore, let $\Upsilon_i\Theta \nearrow_n(\Phi^2_{i_1}\Theta^2_1, \Theta^1_1), \ldots \Upsilon_i\Theta \nearrow_n(\Phi^2_{i_k}\Theta^2_k, \Theta^1_k)$ *exhaustively, and such that* $\Upsilon_i \uparrow^\bullet (\Phi^2_{i_j}, \{\})$ *for each* $0 \leq j \leq k$[8]*. Every subproof ending in* $\vdash \Gamma : \Delta\Phi^1_i \Uparrow {}^n\Upsilon_i\Theta$ *has the following form:*

$$\cfrac{\cfrac{\vdash \Gamma\Phi^2_{i_1}\Theta^2_1 : \Delta\Phi^1_i\Theta^1_1 \Uparrow^\bullet}{\vdots} \quad \cdots \quad \cfrac{\vdash \Gamma\Phi^2_{i_k}\Theta^2_k : \Delta\Phi^1_i\Theta^1_k \Uparrow^\bullet}{\vdots}}{\vdash \Gamma : \Delta\Phi^1_i \Uparrow^n \Upsilon_i\Theta}$$

The figures also describe an if and only if relation: if the premises have cut-free proofs then so does the conclusion.

The lemma for the synchronous phases is the following. The lemma is simpler because there is no longer an unclassified context Θ when a formula is under focus.

Lemma 2. *Let* $R \searrow^n(\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\})$ *and assume that* $\vdash \Gamma : \Delta_1 \Downarrow {}^1b_1, \ldots, \vdash \Gamma : \Delta_m \Downarrow^1 b_m$ *are all cut-free provable and that* $\vdash \Gamma :\Downarrow^\bullet a_1, \ldots, \vdash \Gamma :\Downarrow^\bullet a_n$ *are also cut-free provable. Then* $\vdash \Gamma : \Delta_1 \ldots \Delta_m \Downarrow^n R$ *is also cut-free provable. Furthermore, every cut-free proof of* $\vdash \Gamma : \Delta_1 \ldots \Delta_n \Downarrow^n R$ *is of the form*

$$\cfrac{\cfrac{\vdash \Gamma : \Delta_1 \Downarrow^1 b_1 \;\cdots\; \vdash \Gamma : \Delta_m \Downarrow^1 b_m}{\vdots} \quad \cdots \quad \cfrac{\vdash \Gamma :\Downarrow^\bullet a_1 \;\cdots\; \vdash \Gamma :\Downarrow^\bullet a_n}{\vdots}}{\vdash \Gamma : \Delta_1 \ldots \Delta_m \Downarrow^n R}$$

where $R \searrow^n(\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_m\})$.

The premises with $\Downarrow^1$ must be preceded from above by either $R_1 \Downarrow^1$ or I_1 and the premises with $\Downarrow^\bullet$ are preceded by either $R_2 \Downarrow^\bullet$ or I_2.

These lemmas and the duality of the $\uparrow^n$ and $\downarrow^n$ relations form the basis of the cut-elimination proof.

Since weakening and contraction are not explicit in LUF (except in D_2), we also need the following lemma:

Lemma 3. *If* $\vdash A, A, \Gamma : \Delta \Uparrow^n \Theta$ *has a cut-free proof, then* $\vdash A, \Gamma : \Delta \Uparrow^n \Theta$ *has a cut-free proof of the same height. If* $\vdash \Gamma : \Delta \Uparrow^n \Theta$ *has a cut-free proof, then* $\vdash A, \Gamma : \Delta \Uparrow^n \Theta$ *has a cut-free proof of the same height.*

This lemma is provable by a straightforward induction on the structure of proofs.

We define the *height* of a cut-free proof as the maximum number of alternating asynchronous-synchronous phases (*i.e.*, the number of D_1 and D_2 rules) along a path to a leaf.

We have inherited some syntactic properties of Andreoli's LLF system, among these is the choice to use a singe context or "bin" to the right of the $\Uparrow^n$. This choice simplified our proof system, but also leads to a few additional technicalities. The restricted form of $L\Uparrow^1$ is to ensure that there is no confusion, for example, between decomposing a $\invamp$ and a $\vee^-$. As a result, we cannot always assume that, in a sequent such as $\vdash \Gamma : \Delta :\Uparrow^n A, \Theta$, that Θ can be completely decomposed (absorbed into Γ, Δ) before A can be considered principal. We therefore need the following lemma, which is provable using Lemma 1:

[8] list is no longer unique because of Θ.

Lemma 4. *Let Υ consist of left formulas and let $\Upsilon \uparrow^{\bullet} (\Phi_1^v, \{\}), \ldots, \Upsilon \uparrow^{\bullet} (\Phi_n^v, \{\})$ exhaustively. The sequent $\vdash \Gamma : \Delta \Uparrow^n A, \Upsilon$ has a cut-free proof if and only if $\vdash \Gamma\Phi_i^v : \Delta \Uparrow^n A$ has a cut-free proof of the same proof-height*[9] *for every $1 \leq i \leq n$.*

Theorem 3. *The Cut, cut_1, cut_2 and $cut_\circ$ rules are admissible in LUF.*

Proof. The inductive measure for the cut-elimination proof is the usual lexicographical ordering on the size of the cut formula and the heights of subproofs, where "height" refers to the number of asynchronous-synchronous phases. As usual, we can assume that the two subproofs involved in a cut are cut-free, since we can apply the procedure to the lowest-height cuts first.

Instances of cut are divided into two categories. *Key-case cuts* are cuts where both cut formulas are principal, *i.e.*, when the positive cut formula is under focus and the negative one is being decomposed. *Parametric cuts* refer to cuts when, in at least one subproof, the cut formula is not principal. The *parametric formula* can be synchronous or asynchronous, giving rise to two sub-cases.

We assume without loss of generality that of the two cut formulas A and $A^\perp$, the polarity of A is either +1, +R, or -R. During the initial phase of asynchronous decomposition, the end cut is first permuted to a cut_1, since $R_1 \Uparrow^1$ is applicable to the polarities +1, +R and -R. Our cut-elimination proof is therefore a simultaneous induction on the permutation of cut_1, cut_2 and $cut_\circ$.

The cut-elimination procedure permutes the cut above the introduction of parametric formulas until a key case is reached.

Asynchronous Parametric Decomposition

For the case of asynchronous parametric formulas, Lemma 1 establishes that cut-elimination is invariant under asynchronous decomposition. In particular, the lemma implies that a subproof with negative cut formula $A^\perp$ of the form $\vdash \Gamma' : \Delta' \Uparrow^n A^\perp, \Theta$ will have subproofs ending in $\vdash \Gamma' : \Delta'\Theta^1 \Uparrow^n A^\perp, \Upsilon$, where $\Theta \uparrow^n (\Upsilon, \Theta^1)$. If $A^\perp$ is a -L formula, this will trigger a $L \Uparrow^1$ reaction and Υ can be further decomposed. If $A^\perp$ is a -1 linear formula under $\Uparrow^1$ or a -R formula under $\Uparrow^\circ$ then it can be assumed to be principal as no further decomposition of Υ can occur.

In the special case that $A^\perp$ is a -R formula under $\Uparrow^1$, it will trigger a $R_1 \Uparrow^1$ reaction and enter the linear context Δ, after which Υ can then be completely decomposed, and the cut will permute to a $cut_\circ$.

When parametric asynchronous decomposition reaches the following state:

$$\frac{\vdash \Gamma : \Delta, A \Uparrow^n \quad \vdash \Gamma' : \Delta' \Uparrow^m A^\perp, \Upsilon}{\vdash \Gamma\Gamma' : \Delta\Delta' \Uparrow^1 \Upsilon}\ cut_1 \qquad \text{or} \qquad \frac{\vdash A, \Gamma : \Delta \Uparrow^n \quad \vdash \Gamma' : \Uparrow^m A^\perp, \Upsilon}{\vdash \Gamma\Gamma' : \Delta \Uparrow^n \Upsilon}\ cut_2$$

or

$$\frac{\vdash A, \Gamma : \Delta \Uparrow^n \quad \vdash \Gamma' : \Delta', A^\perp \Uparrow^m}{\vdash \Gamma\Gamma' : \Delta\Delta' \Uparrow^1}\ cut_\circ$$

[9] in terms of the number of decomposition phases.

the left-side subproof (and right-side subproof in the case of $cut_\circ$) must end in a decision rule (D_1 or D_2), which selects a formula for focus. If the formula selected for focus is the cut formula A (and $A^\perp$), then we have a key-case cut. If some other formula in Δ or Γ is selected for focus, then we have a parametric case with a positive parametric formula.

It is also possible that both A and $A^\perp$ are literals, which means that the right-side subproof will also be of the form $\vdash \Gamma' : \Delta', A^\perp \Uparrow$ or $\vdash A^\perp, \Gamma' : \Delta' \Uparrow$, which will then also require a formula to be selected for focus. The negative literal can never be principal. We permute the cut above the subproof that contains the *positive* cut formula. The positive literal is "attractive" in the terminology of [3].

Parametric Focusing

This case is uniform for all three forms of (intermediate) cut. We show that the cut is permuted into multiple cuts of the same type at the borders of the parametric synchronous phases. We assume that all parametric decompositions have been applied to both subproofs of the cut. We demonstrate one principal case:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \Gamma, B : \Delta_1 \Downarrow^1 b_1 \ \cdots \ \vdash \Gamma, B : \Delta_n \Downarrow^1 b_n}{\vdots} \quad \cdots \quad \dfrac{\vdash \Gamma, B : \Downarrow^\bullet a_1 \ \cdots \ \vdash \Gamma, B : \Downarrow^\bullet a_m}{\vdots}}{\vdash \Gamma, B : \Delta, A \Downarrow^\circ B}}{\vdash \Gamma, B : \Delta, A \Uparrow^\bullet} D_2 \qquad \vdash \Gamma' : \Delta' \Uparrow^m A^\perp, \Upsilon}{\vdash \Gamma\Gamma', B : \Delta\Delta' \Uparrow^1 \Upsilon} cut_1$$

where $B \searrow^\circ (\{a_1, \ldots, a_m\}, \{b_1, \ldots, b_n\})$ and $\Delta, A = \Delta_1 \ldots \Delta_n$. This form is guaranteed by Lemma 2. The argument is the same regardless of if B was selected by a D_2 or D_1.

Exactly one of the Δ_i will contain the cut formula A. Let $\Delta_i = \Delta'_i, A$. If b_i is a positive literal, *it cannot be the case that $b_i = A^\perp$ because A is positive.* This critical fact relies on the choice to always permute the cut above the subproof with the positive cut formula when both cut formulas are literals. Thus b_i must be either negative or a non-literal +L formula. By $R_1 \Downarrow^1$, we have a subproof of $\vdash \Gamma, B : \Delta_i \Uparrow^1 b_i$. The original cut is permuted to the following cut:

$$\dfrac{\vdash \Gamma, B : \Delta'_i, A \Uparrow^1 b_i \quad \vdash \Gamma' : \Delta' \Uparrow^m A^\perp, \Upsilon}{\vdash \Gamma\Gamma', B : \Delta'_i\Delta \Uparrow^1 b_i, \Upsilon} cut_1$$

with a lower proof height. Now by Lemma 4, we also have a proof of $\vdash \Phi^\upsilon_j \Gamma\Gamma', B : \Delta'_i\Delta \Uparrow^1 b_i$ for each Φ^υ_j such that $\Upsilon \uparrow^\bullet (\Phi^\upsilon_j, \{\})$. To this we can re-apply *the same* $R_1 \Downarrow^1$rule to obtain $\vdash \Phi^\upsilon_j \Gamma\Gamma', B : \Delta'_i\Delta \Downarrow^1 b_i$. Applying lemma 2 again, we can synthesize the conclusion $\vdash \Phi^\upsilon_j \Gamma\Gamma', B : \Delta\Delta' \Uparrow^1$. Apply Lemma 1 once more to obtain $\vdash \Gamma\Gamma', B : \Delta\Delta' \Uparrow^1 \Upsilon$.

In the special case where A is a -R formula, the cut_1 will eventually permute to a $cut_\circ$.

In the case of cut_2, which means that A is in the classical context, the argument differs as follows. Each premise of the parametric phase is of the form $\vdash B, A, \Gamma$:

$\Delta_i \Downarrow^n b_i$. It is possible that b_i is positive if $\Delta_i = \{b_i^\perp\}$ or if Δ_i is empty and $b_i^\perp \in \Gamma$ (again A is assumed positive if it is a literal). In either case we get by weakening in the form of Lemma 3 that $\vdash B, \Gamma\Gamma'\Phi^v : \Delta_i \Downarrow^n b_i$ is provable. If b_i is negative, then it must be preceded (from above) by $R_1 \Downarrow^1$ or $R_2 \Downarrow^\bullet$. We then permute the cut to a cut_2 between $\vdash B, A, \Gamma : \Delta_i \Uparrow^n b_i$ and $\vdash \Gamma' :\Uparrow^m A^\perp, \Upsilon$, which again gives $\vdash B, \Gamma\Gamma' : \Delta_i \Uparrow^n b^i, \Upsilon$. Again by applying Lemmas 1, 2 and 4, we acquire the conclusion $\vdash B, \Gamma\Gamma' : \Delta \Uparrow^n \Upsilon$.

The parametric cases for $cut_\circ$ are similarly argued. There is in fact no need for Lemma 4 in this case. If the (non-literal) -R cut formula A is selected for focus, it must be that Δ is empty and via $L \Downarrow^1$ and $R_2 \Downarrow^\bullet$, the cut is permuted to a cut_2. When $A^\perp$ is also selected for focus, we have reached the key-cut.

Key Cases

The argument for the key case cuts differ in the cut_1 and cut_2 cases only in that the later involves the permutation of the cut above a contraction. The restricted form of cut_2 is crucial, lest the linear context is illegally copied when the cuts are stacked. We are also aided by the fact that only positive formulas are subject to contraction: A is assumed to be a positive formula in cut_2. The key-case cut is preceded above by several parametric cuts. That is, for the sequent $\vdash A, \Gamma :\Downarrow^\circ A$, the occurrence of A under focus is erased by a key-case cut while the "copy" is erased by parametric cuts. The parametric cuts have lower proof-height measures while the key cut reduces to smaller cut formulas. We detail the argument for the more involved cut_2 case. The case for cut_1 is simpler.

By lemmas 1, 2 and 4, the cut will have the form (after all parametric decompositions)

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash A,\Gamma : \Delta_1 \Downarrow^1 b_1 \cdots \vdash A,\Gamma : \Delta_n \Downarrow^1 b_n}{\vdots} \quad \cdots \quad \dfrac{\vdash A,\Gamma :\Downarrow^\bullet a_1 \cdots \vdash A,\Gamma :\Downarrow^\bullet a_m}{\vdots}}{\vdash A,\Gamma : \Delta \Downarrow^\circ A}}{\vdash A,\Gamma : \Delta \Uparrow^\bullet}\, D_2 \qquad \dfrac{\dfrac{\vdash \Gamma'\Phi_1^2 : \Phi_1^1 \Uparrow^\bullet \Upsilon}{\vdots} \quad \cdots \quad \dfrac{\vdash \Gamma'\Phi_1^2 : \Phi_t^1 \Uparrow^\bullet \Upsilon}{\vdots}}{\vdash \Gamma' :\Uparrow^m A^\perp, \Upsilon}}{\vdash \Gamma\Gamma' : \Delta \Uparrow^\bullet \Upsilon}\, cut_2$$

where $A \searrow^\circ(\{a_1, \ldots, a_m\}, \{b_1, \ldots, b_n\})$, $A^\perp \nearrow_\circ(\Phi_i^2, \Phi_i^1)$ for each $1 \leq i \leq t$ exhaustively and $\Delta = \Delta_1 \ldots \Delta_n$.

By the duality between $\nearrow_n$ and $\searrow^n$, one of the pairs (Φ_k^2, Φ_k^1) will have the form $(\{a_1^\perp, \ldots, a_m^\perp\}, \{b_1^\perp, \ldots, b_n^\perp\})$ The cut can be permuted into zero or more cuts involving formulas of smaller size (or to a single *multicut*) between $\vdash \Gamma'\Phi_k^2 : \Delta'\Phi_k^1 \Uparrow^\bullet$ and the sequents bordering the positive phase on the left subproof.

We reiterate two important invariants of LUF in making these arguments:

- The rule $R_1 \Uparrow^1$ applies to a formula A if and only if $R_1 \Downarrow^1$ or I_1 applies to $A^\perp$.
- The rule $R_2 \Uparrow^\bullet$ applies to A if and only if either $R_2 \Downarrow^\bullet$ or I_2 applies to $A^\perp$.

For clarity in presentation we describe the reduction in stages.

1. If $a_i^\perp$ is a negative literal in Φ_k^2, then the corresponding subproof involving a_i ends with an I_2. This means $a_i \in \Gamma$ because A cannot be a negative literal.

By weakening (in the form of Lemma 3) on the right-side subproof we note that $\vdash \Gamma\Gamma'\Phi^2_k - \{a_i^\perp\} : \Phi^1_k \Uparrow^\bullet \Upsilon$ has a cut-free proof.
Let Φ'^2_k be Φ^2_k without negative literals.

2. If $a_i^\perp$ is positive formula in Φ^2_k, then the corresponding subproof ends $\vdash A, \Gamma :\Uparrow^\circ a_i$, followed by a $R_2 \Downarrow^\bullet$ rule. First we apply a parametric cut to eliminate the extra copy of A that was created when D_2 was applied. Then we form a cut_2 of smaller degree with the right subproof as follows:

$$\dfrac{\vdash \Gamma'\Phi'^2_k : \Phi^1_k \Uparrow^\bullet \Upsilon \qquad \dfrac{\vdash A, \Gamma :\Uparrow^\circ a_i \quad \vdash \Gamma' :\Uparrow^m A^\perp, \Upsilon}{\vdash \Gamma\Gamma' :\Uparrow^\circ a_i, \Upsilon}\ cut_2}{\vdash \Gamma\Gamma'\Gamma'\Phi'^2_k - \{a_i^\perp\} : \Phi^1_k \Uparrow^\bullet \Upsilon\Upsilon}\ cut_2$$

The upper *parametric* cut has a lower proof-height measure than the original and the lower *key-case* cut is made on smaller cut formulas. Both copies of Υ can be absorbed into the classical context and are therefore subject to contraction. That is, by Lemmas 1, 3 and 4, we also have a proof of $\vdash \Gamma\Gamma'\Phi'^2_k - \{a_i^\perp\} : \Phi^1_k \Uparrow^\bullet \Upsilon$.
Note that in a focused sequent calculus there is no need for Gentzen's *mix* rule [4] to handle contraction.
Φ^2_k can now be assumed to be eliminated.

3. If $b_i^\perp$ is a negative literal in Φ^1_k, then the corresponding subproof involving b_i ends with an I_1. This means $\Delta_i = \{b_i^\perp\}$. By weakening on the right-side subproof, we have a cut-free proof with conclusion $\vdash \Gamma\Gamma' : \Delta_i\Phi^1_k - \{b_i^\perp\} \Uparrow^\bullet \Upsilon$.
Let Φ'^1_k be Φ^1_k without negative literals.

4. If $b_j^\perp$ is a positive formula in Φ^1_k, then the corresponding subproof ends $\vdash A, \Gamma : \Delta_i \Uparrow^1 b_j$ followed by a $R_1 \Downarrow^1$ rule. We again form the smaller-measure cuts as follows, this time using cut_1 to reduce the key cut:

$$\dfrac{\vdash \Gamma' : \Delta_{i_1} \ldots \Delta_{i_m}\Phi^{1'}_k \Uparrow^\bullet \Upsilon \qquad \dfrac{\vdash A, \Gamma : \Delta_j \Uparrow^1 b_j \quad \vdash \Gamma' :\Uparrow^m A^\perp, \Upsilon}{\vdash \Gamma\Gamma' : \Delta_j \Uparrow^1 b_j, \Upsilon}\ cut_2}{\vdash \Gamma\Gamma'\Gamma' : \Delta_{i_1} \ldots \Delta_{i_m}\Delta_j\Phi^{1'}_k - \{b_j^\perp\} \Uparrow^1 \Upsilon\Upsilon}\ cut_1$$

Note that the $\Uparrow^1$ in the conclusion is equivalent to $\Uparrow^\bullet$, in the sense that the sequent in one form will have a cut-free proof if and only if the other form has a cut-free proof, by virtue of the $L\Uparrow^1$ rule. Again by Lemmas 1, 3 and 4, we also have a proof of $\vdash \Gamma\Gamma' : \Delta_{i_1} \ldots \Delta_{i_m}\Delta_j\Phi^{1'}_k - \{b_j^\perp\} \Uparrow^\bullet \Upsilon$.
At each successive step, $b_k^\perp$ will be replaced by a Δ_k.

After the sequence of weakenings and cuts we obtain the conclusion $\vdash \Gamma\Gamma' : \Delta \Uparrow^\bullet \Upsilon$.

Our proof of cut-elimination is complete but for the special case when the cut formulas are $\top$ and 0 (and similarly with $\top_l$, 0_r, $\top_r$, and 0_l). One shows that if a 0 can persist in a provable sequent then the same sequent is provable with 0 replaced by anything else. This is a consequence of there being no introduction rule for 0.

Regardless of a cut_1 or cut_2, the key case shows that the essential characteristic of cut-elimination in LUF is to apply cut_2 to the "classically-oriented" subformulas of the original cut-formulas. The linear context (Δ) is necessarily distributed to the subformulas that are treated linearly, i.e., subject to $R_1 \Uparrow^1$ and $R_1 \Downarrow^1$.

A.1 Initial Elimination

Initial elimination tests if the connectives of our logic are *"small enough"*. A core principle of focusing is *when can something be considered a connective.* The $\searrow$ and $\nearrow$ relations used in the cut-elimination proof in fact define the general forms of synthetic introduction rules in LUF. That is to say that the synchronous and asynchronous phases of focused proofs correspond to introduction rules for synthetic connectives.

Theorem 4. *For all formulas A, the sequent $\vdash :\Uparrow^1 A, A^\perp$ is provable in LUF.*

Proof. The result essentially follows from the duality of $\nearrow_n$ and $\searrow^n$. The technical proof is by mutual induction on the following forms:

1. $\vdash \Gamma : A \Uparrow^1 A^\perp$
2. $\vdash A, \Gamma :\Uparrow^\circ A^\perp, \Upsilon$

The case for $\Uparrow^\bullet$ is folded into the others. By Lemma 1, every proof of $\vdash \Gamma : A \Uparrow^1 A^\perp$ has premises of the form $\vdash \Gamma\Phi^2 : \Phi^1, A \Uparrow^\bullet$ such that $A^\perp \nearrow_n(\Phi^2, \Phi^1)$. But then $A \searrow^n(\Phi^{2\perp}, \Phi^{1\perp})$. Choose A to focus on via D_1. By Lemma 2, we can construct a focusing phase ending in either $\vdash \Gamma\Phi^2 : a^\perp \Downarrow^1 a$ or $\vdash \Gamma\Phi^{2\prime}, b^\perp :\Downarrow^\bullet b$ The proofs of these premises follow either by initial rules or by inductive hypotheses. The case for $\vdash A, \Gamma :\Uparrow^\circ A^\perp, \Upsilon$ is similar given Lemma 4.