SUBGROUPS OF FREE METABELIAN GROUPS

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Abstract. In 1954 A. G. Howson proved that the intersection of two finitely generated subgroups of a free group is again finitely generated. Now the free metabelian subgroups of a free metabelian group of finite rank n are quite restricted. Indeed they are again of finite rank at most n. This suggests that there may be an analog of Howson's theorem for free metabelian groups. This turns out not to be the case. The object of this paper is to explore such intersections in free metabelian groups and, more generally, in the wreath product of two free abelian groups. In such a wreath product we show, for instance, that there are algorithms to decide whether or not the intersection of two finitely subgroups is finitely generated or trivial. This leaves open the existence of algorithms to decide the same questions for finitely generated subgroups of finitely generated metabelian groups as a whole.

1. Introduction

1.1. **Finitely generated metabelian groups.** In his ground breaking paper [**?**] in 1954, P. Hall observed that the commutator subgroup $[G, G]$ of a finitely generated metabelian group G can be viewed as a finitely generated module over the integral group ring of the factor derived group $G/[G, G]$. Thus the structure of finitely generated metabelian groups is in large measure determined by the structure of finitely generated modules over polynomial rings in finitely many variables. This enabled Hall to prove a number of beautiful theorems about finitely generated metabelian groups. In particular he showed that they satisfy max-n, the maximal condition for normal subgroups, and hence that there are only a countable number of isomorphism classes of finitely generated metabelian groups. Another consequence of this maximal condition is that the additive group of rational numbers is not a subgroup of a finitely generated metabelian group, which places a restriction on the abelian subgroups of these groups. However, as Hall pointed out in [**?**], the nature of the abelian subgroups remains difficult to determine.

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Hall's work gave rise to a number of positive algorithmic results about finitely generated metabelian groups (see for instance the monograph [?]). In particular, Romanovskii [?] in 1980 proved that there is an algorithm to decide whether or not an element in a finitely generated metabelian group lies in a given finitely generated subgroup. Thus the intersection of two finitely generated subgroups is a recursive set, i.e., there is an algorithm to decide if an element is or is not in the intersection of two finitely generated subgroups. However many problems remain open. Although the word and conjugacy problems have been shown to have positive solutions, the isomorphism problem remains seemingly out of reach at this time. One positive result in this area is the proof by Groves and Miller [?] that there is an algorithm which determines whether or not a finitely generated metabelian group is free metabelian.

It seems that free metabelian groups are more tractable than finitely generated metabelian groups as a whole. For instance it is easy enough to prove that the free metabelian subgroups of a free metabelian group of finite rank n have rank at most n. Indeed a subgroup of a free metabelian group is free if and only if it is generated by a set of elements which are independent modulo the derived group [?]. Moreover, the non-cyclic abelian subgroups of a free metabelian group are contained in the derived group and are therefore free abelian. In fact Wilhelm Magnus [?] has proved that every free metabelian group can be embedded in the wreath product of two free abelian groups and it is easy to prove that the abelian subgroups of such wreath products are free abelian. Many results about finitely generated metabelian groups make use of wreath products and this theorem of Magnus, for example the embedding theorem of Baumslag [?] and Remeslennikov [?]. But these remarks belie the complexity of even this restricted class of metabelian groups. For instance there are continuously many subgroups of the free metabelian group of rank two [?].

It is easy to characterize the finitely generated metabelian groups in which the intersection of finitely generated subgroups are again finitely generated [?]. In general it is not easy to decide whether the intersections of finitely generated subgroups of metabelian groups as a whole are finitely generated. Here we shall prove that there are algorithms which decide for a free metabelian group (or, more generally, for the wreath product of two free abelian groups) whether the intersection of two finitely generated subgroups is finitely generated or trivial. The proof of these results makes use of a new way of describing metabelian groups to which we now turn and which seems to be an essential element in understanding intersections.

1.2. **Hybrid presentations of metabelian groups.** As already noted in the abstract, even the intersection of two finitely generated free metabelian subgroups of a finitely generated free metabelian group need not be finitely generated. We shall give a number of examples of this and related phenomena in Section **??**. In particular we find (Theorem **??**) that the free metabelian group F of rank 2 contains free metabelian subgroups H_1 and H_2 also of rank 2 with intersection $H_1 \cap H_2 = [H_1, H_1] = [H_2, H_2]$. Thus their intersection is their derived group which is a free cyclic $\mathbb{Z}H_i$-module and hence free abelian of countably infinite rank. While this intersection is not finitely generated, it still has a description as a $\mathbb{Z}H_i$-module which is suitably finite. With this and other examples in mind, we introduce the notion of a *hybrid generating system* of a metabelian group.

Definition 1.1. *Let G be a metabelian group. A* hybrid generating system *of G consists of*

1. *an abelian normal subgroup B of G containing the derived group of G;*
2. *a subset X of G which generates G modulo B;*
3. *a subset Y of automorphisms of B which generates an abelian subgroup T of the automorphism group of B;*
4. *a subset Z of B which generates B viewed as a module over the integral group ring $\mathbb{Z}T$ of T.*

Such a hybrid generating system will be termed finite *if X, Y, Z are all finite.*

The notion of a hybrid generating system gives rise in the obvious way to what we term a *hybrid presentation.*

Definition 1.2. *A* hybrid presentation *of a metabelian group G is*

1. *a hybrid generating system B, X, Y, Z of G, as above;*
2. *a presentation of the abelian group T on the generators Y;*
3. *a presentation of B as $\mathbb{Z}T$-module on the generators Z;*
4. *a set of relations R which induce a presentation of G modulo B and take the form $u_\ell = v_\ell$, where the u_ℓ are words in the set X of G and the v_ℓ are $\mathbb{Z}T$-module words in the set Z.*

A hybrid presentation is termed finite *if the sets X, Y, Z of generators in the given hybrid generating system are finite and the corresponding sets of relations are finite.*

Now if G is a metabelian group and B is an abelian normal subgroup of G containing the derived group, then each element $g \in G$ defines an automorphism $\hat{g}$ of B via conjugation:

$$\hat{g} : b \mapsto g^{-1}bg \ (b \in B).$$

The mapping α which sends each element $g \in G$ to $\hat{g}$ is then a homomorphism of G into the automorphism group of B which induces a homomorphism α_* of G/B into the automorphism group of B.

We observe that *every finitely generated metabelian group G has a finite hybrid presentation.* We need first to show that it has a finite hybrid generating system. We choose B to be the derived group of G, X a set of elements of G which generate it modulo its derived group, T the subgroup of the automorphism group of B generated by $Y = \{\alpha_*(x) \mid x \in X\}$ and, since the derived group of a finitely generated metabelian group is finitely generated as a module over the integral group ring of the factor derived group, Z can be chosen to be any such set of generators. This provides us with a finite hybrid presentation of G on choosing a finite presentation for B as a $\mathbb{Z}T$-module and a finite set R of relators which induce a presentation of G modulo B in the manner required.

As another example, observe that if T is a finitely generated free abelian group and B is a free $\mathbb{Z}T$ module with a finite basis, then setting $G = B$, G has a finite hybrid presentation with $X = \emptyset$, Y the free basis of T and Z the module free basis of B, and the set of relations $R = \emptyset$. Now, assuming Y and Z are non-empty, as an abelian group B is free of countably infinite rank. So the same group can have numerous finite hybrid presentations of a very different character.

We remark that the data in a finite hybrid presentation allows one in principle to enumerate a recursive presentation of the underlying metabelian group G as an abstract group.

Our interest in finite hybrid presentations is to use them in investigating the subgroups of given metabelian groups and to help understand how such subgroups intersect. Computing intersections of subgroups is often quite difficult. It turns out that the notion of a hybrid presentation is useful in this connection. To this end, we will need the following definition:

Definition 1.3. *Let G be a metabelian group given by a finite hybrid presentation as above. We term a (not necessarily finitely generated) subgroup H of G* finitely hybrid–presentable *if $H \cap B$ is finitely generated as a P-submodule of B for some subgroup P of T.*

The data given in this definition allows us to find a finite hybrid presentation of H, which explains the terminology. Indeed choose $H \cap B$ to be the appropriate abelian normal subgroup of H as required in

Definition 1.1. Moreover, since G/B is finitely generated, so too is $H/H \cap B$. Consequently the conditions laid down in Definition 1.1 can readily be satisfied. It follows that under these circumstances H has a finite hybrid presentation. We shall make heavy use of this remark in the sequel. Furthermore, it is then not hard to see that a finitely generated subgroup H of a finitely generated metabelian group is finitely hybrid–presentable (see Proposition **??**). We have already observed that groups with a finite hybrid presentation need not be finitely generated.

It is important to note that this discussion of subgroups is relative in the sense that the subgroups here are viewed not in their own right, but as subgroups of the given containing group. For example, consider the free metabelian group F of rank 3 with free generators $\{x_1, x_2, x_3\}$. Let C be the normal closure of $[x_1, x_2]$ which is a free $\mathbb{Z}(F/[F, F])$-module. Consider the subgroup K generated by C and the cyclic group on $[x_1, x_3]$. Then K is not a $\mathbb{Z}P$-submodule for any non-trivial subgroup $P \leq F/[F, F]$. Nor is it finitely generated as a module over the trivial subgroup. So as a subgroup K is not finitely hybrid–presentable. As an abstract group, K is a free abelian group of countably infinite rank, so in several different ways it does have a finite hybrid presentation.

1.3. **Intersections in certain wreath products.** Our analysis of intersections of subgroups in free metabelian groups will apply more generally to certain wreath products. We will make extensive use of the *Magnus embedding* [**?**] of the free metabelian group into the wreath product $W = A \text{ wr } T$ of two finitely generated free abelian groups A and T with bases $\{a_1, \ldots, a_m\}$ and $\{t_1, \ldots, t_n\}$. Thus W is the split extension $W = T \ltimes B$ where B is the free $\mathbb{Z}T$-module with basis $\{a_1, \ldots, a_m\}$. Magnus showed that if F is the free metabelian group with basis $\{x_1, \ldots, x_n\}$ and $m \geq n$ then the map defined by $x_i \mapsto t_i a_i$ is an embedding of F into W.

The wreath product W is of course finitely generated, and W has a finite hybrid presentation as above using the base group B as the abelian normal subgroup. We can now state our main results.

Theorem A. *Let $W = A$ wr T be the wreath product of finitely generated free abelian groups A and T, and let H_1 and H_2 be finitely hybrid–presentable subgroups of W. Then $H_1 \cap H_2$ is finitely hybrid–presentable. Moreover there is a uniform algorithm which, given finite hybrid presentations for H_1 and H_2, computes a finite hybrid presentation for $H_1 \cap H_2$.*

It follows from the proof of Theorem A that it is possible to describe when the intersection of two finitely generated subgroups of such wreath products is again finitely generated:

Theorem B. *Let $W = A \text{ wr } T$ be the wreath product of finitely generated free abelian groups A and T, and let π denote the projection of W onto T. Let H_1 and H_2 be finitely generated subgroups of W and let $H = H_1 \cap H_2$. Then H is finitely generated if and only if either $H \cap B = 1$ or $H\pi$ has finite index in $H_1\pi \cap H_2\pi$.*

As a consequence we can describe algorithms for testing whether such an intersection is finitely generated.

Corollary C. *Let W be the wreath product of two finitely generated free abelian groups. Let H_1 and H_2 be finitely generated subgroups of W. There is a uniform algorithm to determine whether or not $H_1 \cap H_2$ is finitely generated, and, if so, whether or not $H_1 \cap H_2$ is trivial.*

Our methods depend heavily on the fact that the base group B is a free $\mathbb{Z}T$-module. We leave open the question as to whether and how our results can be extended to finitely generated metabelian groups as a whole. Our notion of a finite hybrid presentation is available in the general case, but not all of our results carry over. In Section **??** we give an example showing that Therorem **??** does not carry over to finitely generated metabelian groups in general.

This paper is structured as follows. In Section **??** we set up some notation and review the necessary algorithmic background results. In Section **??** we prove some useful facts about the structure of finitely generated submodules of B and of finitely generated subgroups of W. In Section **??** we show that the intersection $H_1 \cap H_2$ of two subgroups of W with finite hybrid presentations also has a finite hybrid presentation by showing that if $H_i \cap B$ is a finitely generated P_i-module, then $H_1 \cap H_2 \cap B$ is finitely generated as a $P_1 \cap P_2$-module. We also show how to compute a finite set of module generators for $H_1 \cap H_2 \cap B$. In Section **??** we describe an algorithm to compute $(H_1 \cap H_2)\pi$ when H_1 and H_2 are given by finite hybrid presentations, thus completing the calculation of the finite hybrid presentation for $H_1 \cap H_2$ and the proof of Theorem **??**. In Section **??** we characterize those situations in which the intersection of two finitely generated subgroups is itself finitely generated (Theorem **??** and Corollary **??**). In Section **??** we construct a number of examples and use the above results to analyze their properties.

2. Notation and algorithmic background

If G is a group, $\mathbb{Z}G$ denotes the group ring of G over the ring $\mathbb{Z}$ of integers.

In 1954 P. Hall showed that finitely generated abelian-by-polycyclic groups (and hence finitely generated metabelian groups) satisfy max-n, the maximum condition for normal subgroups. Such a group G is an extension of a normal abelian subgroup A by a polycyclic group $P = G/A$. Hall showed that the group ring $\mathbb{Z}P$ is a right noetherian ring and A is a finitely generated $\mathbb{Z}P$-module. This connection between commutative algebra and group theory has been very fruitful and has led to many finiteness conditions and algorithmic results.

As permanent notation we let W denote the wreath product $W = A \operatorname{wr} T$ of two finitely generated free abelian groups A and T with bases $\{a_1, \ldots, a_m\}$ and $\{t_1, \ldots, t_n\}$. The projection of W onto T is denoted by π. We use multiplicative notation in our $\mathbb{Z}T$-modules, so an element $f \in \mathbb{Z}T$ acting on $b \in B$ gives b^f (see below for an example of this notation).

Previously we noted that W is the split extension $W = T \ltimes B$ where B is the free $\mathbb{Z}T$-module with basis $\{a_1, \ldots, a_m\}$. But we will often view B in a slightly different way. Namely B is isomorphic to the restricted direct product of the groups $\{A^t = t^{-1}At \mid t \in T\}$. To see this we observe that any element in the free module B can be written in terms of monomials and hence gives a unique element in the direct product of the groups $\{A^t \mid t \in T\}$. For instance,

$$a_1^{2-5t_1^{-1}t^2} a_2^{3t_1+t_1^{-1}t^2} = (a_1^2)(a_2^3)^{t_1}(a_1^{-5}a_2)^{t_1^{-1}t^2}$$

For $P \leq T$, we will use the term *P-module* for $\mathbb{Z}P$-module. If $\mathcal{S}$ is a subset of the P-module M, we denote by $\operatorname{mod}_P(\mathcal{S})$ the P-submodule of M generated by $\mathcal{S}$. Again we emphasize that we are using multiplicative notation in such P-modules.

Algorithmic questions are of fundamental interest in combinatorial group theory. In the 1950's it was shown that each of Dehn's fundamental decision problems – the word, conjugacy and isomorphism problems for finitely presented groups – is undecidable in general [**?**, **?**]. Then in the mid-1980's it was shown [**?**, **?**] that these problems are also unsolvable for finitely presented solvable groups of derived length at most 3.

The situation is dramatically different, however, if we further restrict to derived length at most 2, i.e. to metabelian groups. In 1959 P. Hall proved that finitely generated metabelian groups are residually finite [**?**]. Since finitely generated metabelian groups satisfy max-n, they are finitely presented in the variety of metabelian groups. It follows that the word problem for such groups is decidable (see, for example, Theorem 9.1.1 in [?]). In 1980 Romanovskii showed the problem of deciding membership in finitely generated subgroups of finitely generated metabelian groups is solvable. Then in 1982 Noskov proved that the conjugacy problem is also decidable in this context [?]. On the other hand, the decidability of the isomorphism problem for finitely presented metabelian groups remains unknown.

Many of the known algorithmic results for metabelian and related solvable groups are collected in the monograph [?], see particularly Sections 9.4 and 9.5. At the heart of many of these algorithms is the fact that the group ring $\mathbb{Z}P$ of a polycyclic group is *submodule computable* in the sense defined in [?]. This means that (1) $\mathbb{Z}P$ is a right Noetherian ring in which the ring operations are computable; (2) every finitely generated right $\mathbb{Z}P$-module M is right Noetherian; and (3) there are algorithms which, when given a finite presentation of M, viewed as a $\mathbb{Z}P$-module, and a finite set S of elements of M, find a presentation for $N = \mod_{\mathbb{Z}P}(S)$ and decide membership in N.

For finitely generated metabelian groups in particular, Baumslag, Cannonito and Robinson [?] demonstrated the decidability of a host of additional natural problems. These include the computation of the derived subgroup, centralizers, the center, the Fitting subgroup and the Frattini subgroup. We will make extensive use of the results and methods of [?] and of [?].

Among the questions left open in [?] are two about the computation of intersections: given a finitely generated metabelian group G and two finitely generated subgroups H and K, can we decide if the intersection $H \cap K$ is finitely generated, and, if so, can we decide if $H \cap K$ is trivial? They show how to answer these questions when at least one of the subgroups, say H, is *nearly normal*, that is, when $H \cap [G, G]$ is normal in G. In this case $H \cap [G, G]$ is finitely generated as a $G/[G, G]$-module. Our results provide a rather different sort of answer in the context of free metabelian groups.

Subgroups with finite hybrid presentations, in our terminology, are similar to these nearly normal subgroups of [?]. Clearly nearly normal subgroups are finitely hybrid–presentable in the case the subgroup B in the definition is the derived group itself, that is, in case $B = [G, G]$. For completeness, we briefly explore the relationship between these two classes of subgroups in the case of the wreath product W.

Let A be free abelian with basis a_1, a_2, a_3, let T be free abelian with basis t_1, t_2, t_3, and let W be the wreath product of A and T. We begin by constructing a subgroup of W which has a finite hybrid presentation but is not nearly normal. Let H be the subgroup of W generated by t_1a_1 and t_2a_2. It is easy to see directly that H has a finite hybrid presentation: since HB/B is free abelian with basis $\{t_1B, t_2B\}$, $H \cap B$ is generated as a normal subgroup of H by the commutator $[t_1a_1, t_2a_2]$; therefore, $H \cap B = H \cap [W, W]$, and it is finitely generated as a $\langle t_1, t_2\rangle$-module. On the other hand, H is not nearly normal since $H \cap [W, W]$ is not normal in W.

Next we construct a subgroup K of W which is nearly normal, but viewed as a subgroup of W, does not have a finite hybrid presentation. Let C be the T-module generated by $[t_1a_1, t_2a_2]$. Let K be the subgroup of W generated by a_3 and C. It is not hard to see that $K = K \cap B$ is not finitely generated as a P-submodule for any $P \leq T$: the only element t of T such that $K^t \leq K$ is $t = 1$, so K is a P-submodule if and only if $P = 1$, but K is not finitely generated as an abelian group. Thus we see that K does not have a finite hybrid presentation. On the other hand, $K \cap [W, W] = C$, and C is finitely generated as a $W/[W, W]$-module, so K is nearly normal.

3. About finite generation

Here we will be concerned with $W = A \text{ wr } T$, where A and T are as usual finitely generated free abelian groups on $\{a_1, \dots, a_m\}$ and $\{t_1, \dots, t_n\}$ and B is the base group of W.

In this section we prove structural theorems about finitely generated modules (not necessarily submodules) contained in B and finitely generated subgroups of W.

For $d \in B$, the *support* of d, denoted by $\sigma(d)$, is the set of all elements $t \in T$ such that the image of d under the projection from B to A^t is non-trivial. Note that $\sigma(d)$ is always finite. If M is a subgroup of B then we put $\sigma(M) = \cup_{d\in M}\sigma(d)$. Observe that if M_1 and M_2 are subgroups of B, then $\sigma(M_1M_2) = \sigma(M_1) \cup \sigma(M_2)$. If $d \in B$ and P is a subgroup of T then the support of the cyclic P-module $d^{\mathbb{Z}P}$ is $\sigma(d)P$. Also $\sigma(M_1 \cap M_2) \subseteq \sigma(M_1) \cap \sigma(M_2)$.

We begin with a lemma characterizing finitely generated modules of B in terms of supports.

Lemma 3.1. *Let P be a subgroup of T, and let $M \leq B$ be a P-module. Then M is finitely generated as a P-module if and only if there exists a finite subset S of T such that $\sigma(M) \subseteq \{sp \mid s \in S, p \in P\}$. If M is generated by $m_1, m_2, \dots, m_k$ as a P-module and $S = \bigcup_{i=1}^{i=k} \sigma(m_i)$, then $\sigma(M) = \{sp \mid s \in S, p \in P\}$.*

Proof. Suppose that M is generated as a P-module by $m_1, \dots, m_k$. Let S be $\bigcup_{i=1}^{i=k} \sigma(m_i)$. For $p \in P$, since $\sigma(m_i^p) = \sigma(m_i)p$, we see that

$\{sp \mid s \in S, p \in P\} \subseteq \sigma(M)$. To see that $\sigma(M) \subseteq \{sp \mid s \in S, p \in P\}$, suppose that $m \in M$. Then m can be written as a product of elements of the form b^p, where $b = m_i^\epsilon$ for some i, some $\epsilon = \pm 1$, and some $p \in P$. Now $\sigma(b^p) \subset SP$, and therefore $\sigma(m) \subset SP$.

For the converse, suppose that there exists a finite subset S of T such that $\sigma(M) \subseteq \{sp \mid s \in S, p \in P\}$. Let $C \leq B$ be the direct product of the subgroups $\{A^{sp} \mid s \in S, p \in P\}$. Notice that C is a P-module. Let $a_1, \ldots, a_n$ be a set of generators for A as an abelian group. C is finitely generated as a P-module by

$$\bigcup_{s \in S} \{a_1^s, \ldots, a_n^s\}.$$

Therefore, C is a Noetherian P-module. Since M is a P-submodule of C, M is finitely generated. □

The next proposition characterizes finitely generated subgroups of W in terms of their intersection with B. Here, as before, π is the projection of W onto T.

Proposition 3.2. *Let H be a subgroup of G and let $P = H\pi$. Then $H \cap B$ is a P-module and H is finitely generated if and only if $H \cap B$ is finitely generated as a P-module.*

Proof. Suppose H is finitely generated. Since $H/(H \cap B)$ is finitely presented, $H \cap B$ is finitely generated as a normal subgroup of H. Therefore, $H \cap B$ is finitely generated as a P-module.

Conversely, suppose that $H \cap B$ is generated as a P-module by $d_1, \ldots, d_r$. Let $h_1, \ldots, h_s$ be elements of H such that $h_1(H \cap B), \ldots, h_s(H \cap B)$ generate $H/(H \cap B)$ as an abelian group. Then $d_1, \ldots, d_r, h_1, \ldots, h_s$ generate H as a group. □

We want to be able to decide whether a subgroup given by a finite hybrid presentation is finitely generated. We begin with an elementary observation about intersections of subgroups of finitely generated abelian groups.

Lemma 3.3. *Let H and K be subgroups of T. Suppose that H is a subset of the union of finitely many cosets of K. Then $[H : H \cap K] < \infty$.*

Proof. Let $h \in H$. Let $\{g_1K, g_2K, \ldots, g_rK\}$ be a set of distinct cosets whose union contains H. By the pigeonhole principle, there exist positive integers $0 < \beta < \alpha \leq r + 1$ and a g_i such that $h^\alpha \in g_iK$ and $h^\beta \in g_iK$. Therefore $h^{\alpha-\beta} \in K$. Therefore, $H/(H \cap K)$ is a finitely generated abelian group of finite exponent, and hence $[H : H \cap K] < \infty$. □

Now we can recognize when a subgroup with a finite hybrid presentation is actually finitely generated.

Proposition 3.4. *Suppose that $H \cap B$ is finitely generated as a P-module for some $P \leq T$. H is finitely generated if and only if $H \cap B = 1$ or $[P : H\pi \cap P] < \infty$.*

Proof. First suppose that H is finitely generated. Then $H \cap B$ is finitely generated as an $H\pi$-module and $H \cap B$ is also finitely generated as a P-module. Therefore, by Proposition **??** there exist finite subsets R and S of T such that $\sigma(H \cap B) = RP = SH\pi$. Fix $r \in R$. For all $p \in P$, there exist $s \in S$ and $q \in H\pi$ such that $sq = rp$, so $p = r^{-1}sq$. Thus, P is a subset of the union of finitely many cosets of $H\pi$. By Lemma **??**, $[P : H\pi \cap P] < \infty$.

Conversely, suppose that $[P : P \cap H\pi] < \infty$ and $H \cap B \neq 1$. Since $H \cap B$ is finitely generated as a P-module, it is also finitely generated as a $P \cap H\pi$-module, and hence as an $H\pi$-module. By Proposition **??**, H is finitely generated as a subgroup. □

4. Intersections

In this section we prove that, in the wreath product W, the intersection of two subgroups with finite hybrid presentations also has a finite hybrid presentation, and we describe an algorithm to find such a description. This, together with Proposition **??**, shows that we can decide whether or not the intersection of two subgroups with finite hybrid presentations is finitely generated, and, if so, if it is trivial.

4.1. **The structure of $H_1 \cap H_2 \cap B$.** Our first proposition is about intersections of submodules of B.

Proposition 4.1. *Let P_1 and P_2 be subgroups of T. Suppose that for $i = 1, 2$, K_i is a finitely generated P_i-submodule of B. Then $K_1 \cap K_2$ is finitely generated as a $P_1 \cap P_2$-module.*

Proof. Let $K = K_1 \cap K_2$. K is a $(P_1 \cap P_2)$-module.

By Lemma **??** it suffices to show that there exists a finite subset U of T such that

$$\sigma(K) \subseteq \{sp \mid s \in U, p \in P_1 \cap P_2\}.$$

By Lemma **??** we know that there exists a finite set R of T such that $\sigma(K_1) = RP_1$. Likewise, there exists a finite set S of T such that $\sigma(K_2) = SP_2$. Since $\sigma(K) \subseteq \sigma(K_1) \cap \sigma(K_2)$, it suffices to show that there exists a finite subset U of T such that $RP_1 \cap SP_2 \subseteq U(P_1 \cap P_2)$.

Let $R = \{r_1 \ldots, r_a\}$ and let $S = \{s_1 \ldots, s_b\}$. Let I be the set of those ordered pairs (i, j) of indices for which there exist $p \in P_1$ and

$q \in P_2$ such that $r_i p = s_j q$. For $(i, j) \in I$, let $p_{i,j} \in P_1$ and $q_{i,j} \in P_2$ be one such solution, so $r_i p_{i,j} = s_j q_{i,j}$. Let

$$U = \{r_i p_{i,j} \mid (i, j) \in I\}.$$

Let $t \in RP_1 \cap SP_2$. Then there exists $(i, j) \in I$, $p \in P_1$, $q \in P_2$ such that $t = r_i p = s_j q$. If we put $z = (p_{i,j})^{-1} p$ then computing in the abelian group T we have

$$z = (p_{i,j})^{-1} p = (r_i p_{i,j})^{-1} r_i p = (s_j q_{i,j})^{-1} s_j q = q_{i,j}^{-1} q$$

Therefore $p_{i,j} p^{-1} \in P_1 \cap P_2$, so $t = r_i p_{i,j} z$ where $z = (p_{i,j})^{-1} p \in P_1 \cap P_2$. □

As a consequence we have the following result which is the first assertion of Theorem **??**. The algorithmic assertions of Theorem **??** will be established in Propositions **??** and **??** below.

Corollary 4.2 (= first part of Theorem **??**). *If H_1 and H_2 are subgroups of W with finite hybrid presentations, then $H_1 \cap H_2$ also has a finite hybrid presentation.*

Proof. Let P_1, P_2 be subgroups of T such that $H_i \cap B$ is finitely generated as a P_i-module. Then $H_1 \cap H_2 \cap B = (H_1 \cap B) \cap (H_2 \cap B)$, which is finitely generated as a $P_1 \cap P_2$-module by Proposition **??**. □

In order to show that we can actually compute $H_1 \cap H_2 \cap B$ we will need to generalize Lemma 2.2 of [**?**] which states that if M is a finitely generated P-module, and M_1 and M_2 are finitely generated P-submodules of M, then we can compute $M_1 \cap M_2$. In particular we will need to be able to compute $M_1 \cap M_2$ when each M_i is a P_i-module, but $P_1 \neq P_2$. Lemma **??** will show that this is possible when M_1 and M_2 are submodules of $B \leq W$; Lemmas **??**, **??** and **??** will pave the way.

Lemma 4.3. *Let G be a finitely generated abelian group, and let H be a subgroup of G. Let F be a free G-module with finite basis $\mathcal{B}$. Let $\mathcal{S}$ be a finite subset of F. There is a uniform algorithm to compute a finite set of H-module generators for $mod_H(\mathcal{B}) \cap mod_G(\mathcal{S})$.*

Proof. This is a special case of Corollary 2.13 of [**?**]. □

Lemma 4.4. *Let $P \leq T$, and let $\mathcal{U} = \{u_1, u_2, \ldots, , u_n\}$ be a set of elements of T such that if $i \neq j$, then $u_i P \neq u_j P$. Then $\mathcal{U}$ is a basis for the free P-submodule N of $\mathbb{Z}T$ given by $N = mod_P U$. Thus, A^{UP} is a free P-submodule with basis $\{a^u \mid a \in \mathcal{A}, u \in \mathcal{U}\}$, where $\mathcal{A}$ is a basis for A as an abelian group.*

Proof. Suppose that $f_1, f_2, \ldots, f_n \in \mathbb{Z}P$ such that $u_1 f_1 + u_2 f_2 + \cdots + u_n f_n = 0$. Notice that $\sigma(u_i f_i) \subset u_i P$, so if $i \neq j$, then $\sigma(u_i f_i)$ is disjoint from $\sigma(u_j f_j)$. From this it follows that $u_i f_i = 0$ for all $i = 1, 2, \ldots, n$. $u_i \in T$ so $u_i \neq 0$: remember, we are using multiplicative notation in T. Since $\mathbb{Z}T$ is an integral domain, it follows that $f_i = 0$ for $i = 1, 2, \ldots, n$. This shows that N is free with basis $\mathcal{U}$.

It is easy to see that since $\mathcal{U}$ is a basis for a free P-submodule N of $\mathbb{Z}T$, then $A^N = A^{UP}$ is a free P-submodule of B with basis $\{a^u \mid a \in \mathcal{A}, u \in \mathcal{U}\}$. □

Lemma 4.5. *Let W be the wreath product of finitely generated free abelian groups A and T. Let P and P_1 be subgroups of T such that $P \leq P_1$. Let M_1 be a finitely generated P_1-submodule of B. Let U be a finite subset of T such that if $u, v \in U$, then $uP_1 \neq vP_1$. Then there is a uniform algorithm to compute a finite set of generators for $M_1 \cap mod_P A^U$ as a P-module.*

Proof. Let $Q = UP_1 \cup \sigma(M_1)$. By Lemma **??**, $\sigma(M_1)$ is the union of finitely many cosets of P_1. Therefore, Q is itself the union of finitely many cosets of P_1. Thus there exists a finite subset $\mathcal{Q} \supseteq U$ of T such that $Q = \bigcup_{u \in \mathcal{Q}} uP_1$ and $u, v \in \mathcal{Q}$, $u \neq v$ implies that $uP_1 \neq vP_1$. Let $\mathcal{C} = \{a^q \mid a \in \mathcal{A}, q \in \mathcal{Q}\}$. By Lemma **??** A^Q is free as a P_1-module and $\mathcal{C}$ is a basis for A^Q as such.

Let N be the P-module given by $N = \mathrm{mod}_P \mathcal{C} \cap M_1$. By Lemma **??** we can find generators for N as a P-module. By Lemma 2.2 of [**?**] we can then compute $N' = N \cap \mathrm{mod}_P A^U$ as a P-module.

We will now show that $N' = M_1 \cap \mathrm{mod}_P A^U$. Since $U \subseteq \mathcal{Q}$, $\mathrm{mod}_P A^U \subseteq \mathrm{mod}_P \mathcal{C}$. Therefore $M_1 \cap \mathrm{mod}_P A^U \subseteq M_1 \cap \mathrm{mod}_P \mathcal{C} = N$ and hence $M_1 \cap \mathrm{mod}_P A^U \subseteq N'$. On the other hand, $N' \subseteq N \subseteq M_1$, and clearly $N' \subseteq \mathrm{mod}_P A^U$. Therefore $N' \subseteq M_1 \cap \mathrm{mod}_P A^U$. □

Lemma 4.6. *Let W be the wreath product of finitely generated free abelian groups A and T. For $i = 1, 2$, let P_i be a subgroup of T, and let $P = P_1 \cap P_2$. Let M_i be a finitely generated P_i-submodule of B and let M be a finitely generated P-module such that $M \geq M_1 \cap M_2$. Then there is a uniform algorithm to compute a finite set of generators for $M_1 \cap M_2$ as a P-module.*

Proof. It suffices to describe an algorithm to compute P-module generators for $M \cap M_1$ since $M_1 \cap M_2 = (M \cap M_1) \cap (M \cap M_2)$. The difficulty arises because M_1 may not be finitely generated as a P-module. In the proof of Proposition **??**, we see that we can compute a finite subset U of T such that $\sigma(M_1 \cap M) \subseteq UP$ and $u, v \in U$, $u \neq v$ implies $uP \neq vP$.

Let $N = \text{mod}_P A^U$. Then $M_1 \cap M \subseteq N$. It is sufficient to find a finite set of generators for $N \cap M_1$ as a P-module, since we can then compute $M \cap M_1 = M \cap (N \cap M_1)$. Lemma **??** suffices for computing $N \cap M_1$. □

The next result provides one of the algorithms needed for Theorem **??**. The other required algorithm is given by Proposition **??** below.

Proposition 4.7. *Let W be the wreath product of finitely generated free abelian groups A and T and let H_1 and H_2 be subgroups of W given by finite hybrid presentations. There is a uniform algorithm to compute a finite generating set for $H_1 \cap H_2 \cap B$ as an $H_1\pi \cap H_2\pi$-module.*

Proof. Suppose that $H_i \cap B$ is given as a P_i-module. Let $P = P_1 \cap P_2$. Our first task to find a finite set $U \subseteq T$ such that $H_1 \cap H_2 \cap B$ is contained in the P-module generated by the direct sum of $\{A^u \mid u \in U\}$. Examination of the proof of Proposition **??** shows that given elements $r, s \in T$, we must be able to decide if there exist elements $p_1 \in P_1$ and $p_2 \in P_2$ such that $rp_1 = sp_2$, and to find such elements if they exist. Since $rp_1 = sp_2$ if and only if $rs^{-1} = p_1^{-1}p_2$, we can use linear algebra to test whether or not rs^{-1} is an element of P_1P_2. If it is, we can enumerate the elements of P_1 and P_2 until we find $p_1 \in P_1$ and $p_2 \in P_2$ such that $rs^{-1} = p_1^{-1}p_2$. By Lemma **??** we can then compute a finite generating set for $H_1 \cap H_2 \cap B$. □

4.2. **The structure of $(H_1 \cap H_2)\pi$.** Suppose we are given finite hybrid presentations for subgroups H_1 and H_2 of W. The finite hybrid presentation for H_i consists of a finite set of elements of H_i whose images generate H_iB/B, a finite set of elements of T that generate a subgroup we call P_i, and a finite set of elements of H_i that generate $H_i \cap B$ as a P_i-module. In order to complete our calculation of a finite hybrid presentation for $H_1 \cap H_2$, we must compute $(H_1 \cap H_2)\pi$. Let $Q_i = H_iB/B$. Note that while $H_i \cap B$ is also a Q_i-module, it may not be finitely generated as such.

We examine three cases:

(1) $B \leq H_2$;
(2) $Q_1 = Q_2$;
(3) the general case.

Case 1 is easy: if $B \leq H_2$, then $(H_1 \cap H_2)B = H_1B \cap H_2B$, so computations in W/B suffice to find generators for $(H_1 \cap H_2)B/B$.

It is also easy to see that the general case (3) can be handled assuming that Cases 1 and 2 can be handled. To see this, let $K = H_1B \cap H_2B$.

Computations in W/B suffice for computing K. Let $H_i' = H_i \cap K$. Using Case 1 we compute generators for $H_i'B/B$. Now $H_1' \cap H_2' = H_1 \cap H_2$. Furthermore,

$$\begin{aligned} H_1'B &= (H_1 \cap (H_1B \cap H_2B))B \\ &= (H_1 \cap H_2B)B \\ &= H_1B \cap H_2B. \end{aligned}$$

Therefore $H_1'B = H_2'B$, so we use Case 2 to compute $(H_1' \cap H_2')B/B = (H_1 \cap H_2)B/B$ as required.

We are left with Case 2, in which $Q_1 = Q_2$. Let $Q = Q_1$ (so $Q = Q_2$ as well). Notice that $H_1 \cap B$ and $H_2 \cap B$ are both Q-modules, though they might not be finitely generated as such.

Our primary task will be to find generators for the following subgroup P of Q:

$$P = \{p \in Q \mid \exists a \in B \; s.t. \; pa \in H_1 \cap H_2\}$$

We will now mimic the proof of Theorem 5.6 of [?] by describing P as the kernel of a derivation δ from Q to a quotient of a certain Q-submodule of B. We will do so in several steps.

In this case there exist sets $S_1 \subseteq H_1$ and $S_2 \subseteq H_2$ of the following form:

$$\begin{aligned} S_1 &= \{x_1b_1, x_2b_2, \ldots, x_rb_r\} \\ S_2 &= \{x_1c_1, x_2c_2, \ldots, x_rc_r\} \end{aligned}$$

where the x_i's form a basis for Q as a free abelian group, the b_i's and c_i's are in B, and where the images of S_i generate H_iB/B.

Let θ_i be the map that takes $m \in \mathbb{Z}$ to $f \in \mathbb{Z}Q$ defined as follows: for any $b \in B$, $(x_ib)^m = x_i^m b^f = x_i^m b^{m\theta_i}$. For $m > 0$, one can check that $m\theta_i = 1 + x_i + \ldots + x_i^{m-1}$ and that $(-m)\theta_i = -x_i^{-1} - x_i^{-2} - \ldots - x_i^{-m}$. Consequently, for all $m \in \mathbb{Z}$, the identity $(x_i - 1)(m\theta_i) = (x_i^m - 1)$ holds. We also define the elements

$$d_i = (x_ic_i)^{-1}(x_ib_i) = b_ic_i^{-1}$$

for $i = 1, 2, \ldots, r$, which measure the difference between the corresponding generators of the S_i.

Lemma 4.8.

$$(x_1b_1)^{v_1}(x_2b_2)^{v_2}\cdots(x_rb_r)^{v_r} = x_1^{v_1}x_2^{v_2}\cdots x_r^{v_r} b_1^{v_1\theta_1 x_2^{v_2}x_3^{v_3}\cdots x_r^{v_r}} b_2^{v_2\theta_2 x_3^{v_3}x_4^{v_4}\cdots x_r^{v_r}} \cdots b_r^{v_r\theta_r}.$$

Similar equations hold for the c_i and d_i

Proof.

$$(x_1b_1)^{v_1}(x_2b_2)^{v_2}\cdots(x_rb_r)^{v_r} = (x_1^{v_1}b_1^{v_1\theta_1})(x_2^{v_2}b_2^{v_2\theta_2})\cdots(x_r^{v_r}b_r^{v_r\theta_r})$$
$$= x_1^{v_1}x_2^{v_2}\cdots x_r^{v_r}b_1^{v_1\theta_1x_2^{v_2}x_3^{v_3}\cdots x_r^{v_r}}b_2^{v_2\theta_2x_3^{v_3}x_4^{v_4}\cdots x_r^{v_r}}\cdots b_r^{v_r\theta_r}$$

□

We now define a set map μ from Q to B by

$$\mu : x_1^{v_1}x_2^{v_2}\cdots x_r^{v_r} \mapsto d_1^{v_1\theta_1x_2^{v_2}x_3^{v_3}\cdots x_r^{v_r}}d_2^{v_2\theta_2x_3^{v_3}x_4^{v_4}\cdots x_r^{v_r}}\cdots d_r^{v_r\theta_r}$$

Notice that by Lemma **??**, if $q = x_1^{v_1}x_2^{v_2}\cdots x_r^{v_r}$, then

$$q\mu = q^{-1}(x_1d_1)^{v_1}(x_2d_2)^{v_2}\cdots(x_rd_r)^{v_r}.$$

That is, if we substitute x_id_i for x_i in q and express the result in the form qb with $b \in B$, then $b = q\mu$. Alternatively, if we use functional notation $q = q(x_i)$ and $q(x_ib_i)$ and $q(x_ic_i)$ are the results of replacing the x_i by x_ib_i and x_ic_i respectively, then $q\mu = q(x_ic_i)^{-1}q(x_ib_i)$. The following proposition gives a key property of this map μ for intersections.

Proposition 4.9. *$q \in P$ if and only if $q\mu \in (H_1 \cap B)(H_2 \cap B)$.*

Proof. We have the following chain of equivalences:

$$x_1^{v_1}x_2^{v_2}\cdots x_r^{v_r} \in P$$

$$
\begin{aligned}
&\Leftrightarrow \quad \exists a \in B \; s.t. \; x_1^{v_1} x_2^{v_2} \cdots x_r^{v_r} a \in H_1 \cap H_2 \\
&\Leftrightarrow \quad \exists a \in B \; s.t. \\
&\qquad (x_1^{v_1} x_2^{v_2} \cdots x_r^{v_r} a)^{-1} (x_1 b_1)^{v_1} (x_2 b_2)^{v_2} \cdots (x_r b_r)^{v_r} \in H_1 \cap B \text{ and} \\
&\qquad (x_1^{v_1} x_2^{v_2} \cdots x_r^{v_r} a)^{-1} (x_1 c_1)^{v_1} (x_2 c_2)^{v_2} \cdots (x_r c_r)^{v_r} \in H_2 \cap B \\
&\Leftrightarrow \quad \exists a \in B, m_1 \in H_1 \cap B, m_2 \in H_2 \cap B \; s.t. \\
&\qquad a^{-1} b_1^{v_1 \theta_1 x_2^{v_2} x_3^{v_3} \cdots x_r^{v_r}} b_2^{v_2 \theta_2 x_3^{v_3} x_4^{v_4} \cdots x_r^{v_r}} \cdots b_r^{v_r \theta_r} = m_1 \text{ and} \\
&\qquad a^{-1} c_1^{v_1 \theta_1 x_2^{v_2} x_3^{v_3} \cdots x_r^{v_r}} c_2^{v_2 \theta_2 x_3^{v_3} x_4^{v_4} \cdots x_r^{v_r}} \cdots c_r^{v_r \theta_r} = m_2 \\
&\Leftrightarrow \quad \exists a \in B, m_1 \in H_1 \cap B, m_2 \in H_2 \cap B \; s.t. \\
&\qquad a = b_1^{v_1 \theta_1 x_2^{v_2} x_3^{v_3} \cdots x_r^{v_r}} b_2^{v_2 \theta_2 x_3^{v_3} x_4^{v_4} \cdots x_r^{v_r}} \cdots b_r^{v_r \theta_r} m_1 \text{ and} \\
&\qquad a = c_1^{v_1 \theta_1 x_2^{v_2} x_3^{v_3} \cdots x_r^{v_r}} c_2^{v_2 \theta_2 x_3^{v_3} x_4^{v_4} \cdots x_r^{v_r}} \cdots c_r^{v_r \theta_r} m_2 \\
&\Leftrightarrow \quad \exists m_1 \in H_1 \cap B, m_1 \in H_2 \cap B \; s.t. \\
&\qquad b_1^{v_1 \theta_1 x_2^{v_2} x_3^{v_3} \cdots x_r^{v_r}} b_2^{v_2 \theta_2 x_3^{v_3} x_4^{v_4} \cdots x_r^{v_r}} \cdots b_r^{v_r \theta_r} m_1 = \\
&\qquad c_1^{v_1 \theta_1 x_2^{v_2} x_3^{v_3} \cdots x_r^{v_r}} c_2^{v_2 \theta_2 x_3^{v_3} x_4^{v_4} \cdots x_r^{v_r}} \cdots c_r^{v_r \theta_r} m_2 \\
&\Leftrightarrow \quad \exists m_1 \in H_1 \cap B, m_2 \in H_2 \cap B \; s.t. \\
&\qquad d_1^{v_1 \theta_1 x_2^{v_2} x_3^{v_3} \cdots x_r^{v_r}} d_2^{v_2 \theta_2 x_3^{v_3} x_4^{v_4} \cdots x_r^{v_r}} \cdots d_r^{v_r \theta_r} = m_1^{-1} m_2
\end{aligned}
$$

which is the desired result. □

Lemma 4.10. *For all* $i, j = 1, 2, \ldots, r$,

$$[x_i d_i, x_j d_j] \in (H_1 \cap B)(H_2 \cap B).$$

Proof. Since T is abelian and so $[x_i, x_j] = 1$ an easy calculation gives

$$[x_i d_i, x_j d_j] = d_i^{-1} x_i^{-1} d_j^{-1} x_j^{-1} x_i d_i x_j d_j = d_i^{x_j - 1} d_j^{1 - x_i}$$

We get a similar equation by substituting b's for d's, and another by substituting c's for d's. Since $d_i^{x_j-1} d_j^{1-x_i} = b_i^{x_j-1} b_j^{1-x_i} (c_i^{x_j-1} c_j^{1-x_i})^{-1}$, $[x_i d_i, x_j d_j] = [x_i b_i, x_j b_j]([x_i c_i, x_j c_j])^{-1} \in (H_1 \cap B)(H_2 \cap B)$. □

Let M be the Q-submodule of B generated by the d_i's and the Q-module $(H_1 \cap B)(H_2 \cap B)$. Let δ be the map from Q to $M/(H_1 \cap B)(H_2 \cap B)$ given by $q\delta = q\mu(H_1 \cap B)(H_2 \cap B)$.

Lemma 4.11. *δ is a derivation.*

Proof. Let $p = x_1^{u_1} x_2^{u_2} \cdots x_r^{u_r}$, and let $q = x_1^{v_1} x_2^{v_2} \cdots x_r^{v_r}$. Then modulo $(H_1 \cap B)(H_2 \cap B)$ we have the following congruences

$$\begin{aligned} (pq)\delta &\equiv (pq)^{-1}(x_1 d_1)^{u_1+v_1}(x_2 d_2)^{u_2+v_2} \cdots (x_r d_r)^{u_r+v_r} \\ &\equiv (pq)^{-1}(x_1 d_1)^{u_1}(x_1 d_1)^{v_1} \cdots (x_r d_r)^{u_r}(x_r d_r)^{v_r} \\ &\equiv (pq)^{-1}(x_1 d_1)^{u_1} \cdots (x_r d_r)^{u_r}(x_1 d_1)^{v_1} \cdots (x_r d_r)^{v_r} \\ &\equiv (p^{-1}(x_1 d_1)^{u_1} \cdots (x_r d_r)^{u_r})^q q^{-1}(x_1 d_1)^{v_1} \cdots (x_r d_r)^{v_r} \\ &\equiv (p\delta)^q (q\delta). \end{aligned}$$

(Notice the use of Lemma **??** in line 3.) That is, $(pq)\delta \equiv (p\delta)^q(q\delta)$ mod $(H_1 \cap B)(H_2 \cap B)$. □

This brings us to the following result which is the remaining algorithm needed for Theorem **??**. Taken together, Corollary **??** and Propositions **??** and **??** establish Theorem **??**.

Proposition 4.12. *Let $W = A$ wr T be the wreath product of finitely generated free abelian groups A and T, and let H_1 and H_2 be subgroups of W given by finite hybrid presentations, where $H_i \cap B$ is given as a P_i-module. There is a uniform algorithm to compute a finite set of generators for $(H_1 \cap H_2)\pi$.*

Proof. As we saw at the start of this section, we may assume that $H_1\pi = H_2\pi$. Let $Q = H_1\pi$. Let δ be the derivation from Q to $M/(H_1 \cap B)(H_2 \cap B)$ defined above. By Proposition **??**, the kernel of δ is P. Clearly, $M/(H_1 \cap B)(H_2 \cap B)$ is finitely generated as a Q-module, so by Lemma 5.5 of [**?**], we can compute P. Since membership testing in $H_1 \cap H_2$ is possible, for each generator p of P we can do an exhaustive enumeration search to find an element b of B such that $pb \in H_1 \cap H_2$. The images in $W\pi$ of the elements $pb \in H_1 \cap H_2$ so obtained generate $(H_1 \cap H_2)\pi$. □

4.3. **Finite generation of intersections.** Notice that $(H_1 \cap H_2)\pi \leq H_1\pi \cap H_2\pi$, but the reverse inequality does not necessarily hold; indeed, $(H_1 \cap H_2)\pi$ may not have finite index in $H_1\pi \cap H_2\pi$. The proposition below shows that this fact is at the heart of why the intersection of two finitely generated subgroups of W may not itself be finitely generated.

Theorem 4.13 (= Theorem **??**)**.** *Let H_1 and H_2 be finitely generated subgroups of W and let $H = H_1 \cap H_2$. Then H is finitely generated if and only if either $H \cap B = 1$ or $H\pi$ has finite index in $H_1\pi \cap H_2\pi$.*

Proof. Let $K = H \cap B$. By Proposition **??** H is finitely generated if and only if K is finitely generated as an $H\pi$-module.

Suppose that H is finitely generated and that $K \neq 1$. By Lemma **??** there exists a nonempty finite subset S of T such that $\sigma(K) \subseteq \{sp \mid s \in S, p \in H\pi\}$. But K is a $(H_1\pi \cap H_2\pi)$-module, so $\sigma(K)$ must be closed under right multiplication by $H_1\pi \cap H_2\pi$. It follows that $H\pi$ must have finite index in $H_1\pi \cap H_2\pi$.

Clearly, if $K = 1$, then H is finitely generated since it is isomorphic to a subgroup of T. So assume that $K \neq 1$ and that $H\pi$ has finite index in $H_1\pi \cap H_2\pi$. By Proposition **??** K is finitely generated as a $(H_1\pi \cap H_2\pi)$-module. Since $H\pi$ has finite index in $H_1\pi \cap H_2\pi$, K is also finitely generated as a $H\pi$-module. □

This leads to the following decidability result:

Corollary 4.14 (= Corollary **??**). *Let W be the wreath product of two finitely generated free abelian groups. Let H_1 and H_2 be finitely generated subgroups of W. There is a uniform algorithm to determine whether or not $H_1 \cap H_2$ is finitely generated, and, if so, whether or not $H_1 \cap H_2$ is trivial.*

Proof. We can compute finite hybrid presentations for H_1 and H_2, and from these, by Propositions **??** and **??**, we can compute a finite hybrid presentation for $H_1 \cap H_2$. From this finite hybrid presentation it is obvious whether $H_1 \cap H_2$ are trivial since this is the case if and only if $H_1 \cap H_2 \cap B$ and $(H_1 \cap H_2)\pi$ are both trivial. If $H_1 \cap H_2 \cap B$ is trivial, then $H_1 \cap H_2$ is finitely generated. If $H_1 \cap H_2 \cap B$ is not trivial, then $H_1 \cap H_2$ is finitely generated if and only if $(H_1 \cap H_2)\pi$ has finite index in $H_1\pi \cap H_2\pi$, and this is easy to decide using linear algebra. □

5. Howson's theorem and some examples of intersections of finitely generated metabelian groups

We again recall Howson's Theorem [?] from 1954: *if F is a free group. and H_1 and H_2 are finitely generated subgroups of F, then $H_1 \cap H_2$ is finitely generated.*

Now instead let F be a finitely generated free metabelian group. It is not difficult to produce two finitely generated subgroups of F whose intersection is not finitely generated (one such example is provided below), so the obvious analog of Howson's Theorem does not hold in this context. It is natural to then ask: which subgroups of a free metabelian group are more closely analogous to finitely generated subgroups of a free group? All subgroups of free groups are free, suggesting that there is an analog of Howson's theorem for finitely generated free metabelian subgroups of F. In any group, the intersection of two finite index subgroups is finitely generated by virtue of being of finite index itself.

Another possibility then is to restrict attention to those free metabelian subgroups which are of finite index modulo the derived group of the given supergroup. Here we give one example that illustrates that Howson's Theorem does not carry over to free metabelian groups in either instance.

We construct two finitely generated subgroups of the free metabelian group F that are themselves free, whose projections in $F/[F,F]$ generate all of $F/[F,F]$, and whose intersection is not finitely generated.

We again remark that the free metabelian subgroups of a finitely generated free metabelian group are of restricted rank. Specifically, if F_n is the free metabelian group of rank n, then F_n can not be embedded in F_{n-1}, the free metabelian group of rank $n-1$. In fact, more generally it can be shown that the free solvable groups $F_n(S_\ell)$ of rank n and derived length ℓ cannot be embedded in a finite direct power of $F_{n-1}(S_\ell)$ for any n (see Corollary 25.73 in [**?**]).

Theorem 5.1. *Let F be the free metabelian group of rank 2. Then there exist finitely generated free metabelian subgroups H_1 and H_2 of F such that $H_1[F,F] = H_2[F,F] = F$, but $H_1 \cap H_2$ is not finitely generated. Moreover, $H_1 \cap H_2 = H_1 \cap B = H_2 \cap B$ and is a free cyclic $\mathbb{Z}(F/[F,F])$-module.*

Proof. Let W be the wreath product of two finitely generated free abelian groups A and T, each of rank 2 with bases a_1, a_2 and t_1, t_2 respectively. We adopt our usual notation: B is the direct product of $\{A^t \mid t \in T\}$ that is normal in W and π is the projection of W onto $T = W/B$.

Let F be the subgroup group of W that is generated by t_1a_1 and t_2a_2. F is free metabelian [**?**]. Let a be the commutator of the generators for F, so $a = a_1^{t_2-1}a_2^{1-t_1}$. $F \cap B$ equals $[F,F]$ and is generated as a T-module by a. Let $H_1 = \langle t_1a_1a^{t_1}, t_2a_2a^{t_2}\rangle$ and let $H_2 = \langle t_1a_1a, t_2a_2a\rangle$. Clearly H_1 and H_2 are subgroups of F and $H_1\pi = H_2\pi = W\pi = T$. Furthermore $H_1[F,F] = H_2[F,F] = F$.

We will first show that $H_1 \cap B = H_2 \cap B \neq 1$. Let b be the commutator of the generators of H_1. Then $H_1 \cap B$ is generated as a T-module by b where

$$\begin{aligned} b &= [t_1a_1a^{t_1}, t_2a_2a^{t_2}] \\ &= (a_1a^{t_1})^{t_2-1}(a_2a^{t_2})^{1-t_1} \\ &= a^{1+t_1(t_2-1)+t_2(1-t_1)} \\ &= a^{1-t_1+t_2}. \end{aligned}$$

But $H_2 \cap B$ is generated as a T-module by b as well since

$$[t_1a_1a, t_2a_2a] = (a_1a)^{t_2-1}(a_2a)^{1-t_1} = a^{1+(t_2-1)+(1-t_1)} = a^{1-t_1+t_2}.$$

Thus $H_1 \cap B = H_2 \cap B \neq 1$ as claimed.

Since $H_1\pi = H_2\pi = T$ and $H_1 \cap B = H_2 \cap B$ is the free cyclic $\mathbb{Z}T$-module generated by $b = a^{1-t_1+t_2}$, it follows that H_1 and H_2 are free metabelian groups [**?**]. Note that the free cyclic module $H_1 \cap B$ is not finitely generated as a group (it is free abelian of countably infinite rank).

Finally, we will show that $(H_1 \cap H_2) \leq B$, and hence that $H_1 \cap H_2 = H_1 \cap B$ which we have already calculated. We define d_1, d_2 as in Section **??**, so

$$d_i = (t_ia_ia)^{-1}(t_ia_ia^{t_i}) = (a_ia^{t_i})(a_ia)^{-1} = a^{t_i-1}.$$

Let n_1 and n_2 be integers such that $t_1^{n_1}t_2^{n_2} \in (H_1 \cap H_2)\pi$. Then by Proposition **??** we know $d_1^{(n_1\theta_1)t_2^{n_2}} d_2^{n_2\theta_2} = a^{(1-t_1+t_2)f}$ for some $f \in \mathbb{Z}T$. Therefore

$$\begin{aligned}(t_1-1)(n_1\theta_1)t_2^{n_2} + (t_2-1)(n_2\theta_2) &= (1-t_1+t_2)f \\ (t_1^{n_1}-1)t_2^{n_2} + (t_2^{n_2}-1) &= (1-t_1+t_2)f\end{aligned}$$

and hence we have

$$t_1^{n_1}t_2^{n_2} - 1 = (1-t_1+t_2)f.$$

Such an equation is only possible if $n_1 = n_2 = 0$ (and $f = 0$). To see this observe that $\mathbb{Z}T$ is naturally embedded in $\mathbb{Q}T$. The retraction from $\mathbb{Q}T$ to $\mathbb{Q}$ defined by $t_1 \mapsto 3$ and $t_2 \mapsto 2$ sends the right hand side of the equation to 0. But the left hand side is sent to $3^{n_1}2^{n_2} - 1$ which is 0 only when $n_1 = n_2 = 0$, as claimed. Thus $(H_1 \cap H_2) \leq B$, completing the proof of the theorem. □

If H_1 and H_2 are subgroups satisfying the hypotheses of Proposition **??**, then is it always the case that $H_1 \cap H_2$ is not finitely generated? Our final example shows that the answer to this question is "No".

Proposition 5.2. *Let F be the free metabelian group of rank 2. Then there exist finitely generated free metabelian subgroups H_1 and H_2 of F $H_1[F,F] = H_2[F,F] = F$, $H_1 \not\leq H_2$, $H_2 \not\leq H_1$, and $H_1 \cap H_2$* is *finitely generated.*

Proof. Let W be the wreath product of two finitely generated free abelian groups A and T, each of rank 2 with bases a_1, a_2 and t_1, t_2 respectively. We adopt our usual notation: B is the direct product of

$\{A^t \mid t \in T\}$ that is normal in W; π is the projection of W onto W/B; $\mathbb{Z}T$ is the group ring of T.

Let F be the subgroup group of W that is generated by t_1a_1 and t_2a_2. F is free metabelian by [**?**]. Let a be the commutator of the generators for F, so $a = a_1^{t_2-1}a_2^{1-t_1}$. Note that $F \cap B$ equals $[F, F]$ and is generated as a $\mathbb{Z}T$-module by a.

Let $H_1 = \langle t_1a_1a^{t_1}, t_2a_2a^{t_1}\rangle$ and let $H_2 = \langle t_1a_1a, t_2a_2a\rangle$. It is clear that $H_1\pi = H_2\pi = W\pi = T$ and also that $H_1[F, F] = H_2[F, F] = F$. We will see below that $H_1 \cap B$ and $H_2 \cap B$ are nontrivial, from which it follows [**?**] that H_1 and H_2 are free metabelian with bases the given generators. It remains to prove that $H_1 \cap H_2$ *is* finitely generated and that $H_1 \not\leq H_2$ and $H_2 \not\leq H_1$.

We will begin by showing that $H_1 \cap H_2$ is finitely generated. By Proposition **??** we know $H_1 \cap H_2 \cap B$ is finitely generated as a T-module. So it suffices to show that $(H_1 \cap H_2)\pi = T$. Then by Proposition **??** it suffices to show that $t_1\mu = d_1$ and $t_2\mu = d_2$ both belong to $(H_1 \cap B)(H_2 \cap B)$. In the present example $d_1 = a_1a^{t_1}(a_1a)^{-1}$ and $d_2 = a_2a^{t_1}(a_2a)^{-1}$, and so $d_1 = d_2 = a^{t_1-1}$.

Now $H_1 \cap B$ is generated as a T-module by

$$\begin{aligned}[t_1a_1a^{t_1}, t_2a_2a^{t_1}] &= a^{-t_1}a_1^{-1}t_1^{-1}a^{-t_1}a_2^{-1}t_2^{-1}t_1a_1a^{t_1}t_2a_2a^{t_1}\\ &= a_1^{t_2-1}a_2^{1-t_1}a^{-t_1^2+t_1t_2}\\ &= a^{1-t_1^2+t_1t_2}\end{aligned}$$

Therefore $H_1 \cap B$ consists of all elements B of the form a^f where f is in the ideal $\mathcal{I}$ of $\mathbb{Z}T$ generated by $1 - t_1^2 + t_1t_2$. A similar calculation shows that $[t_1a_1a, a, t_2a_2a] = a^{1-t_1+t_2}$. Therefore $H_2 \cap B$ is generated by $a^{1-t_1+t_2}$, and hence $H_2 \cap B$ consists of all elements B of the form a^f where f is in the ideal $\mathcal{J}$ of $\mathbb{Z}T$ generated by $1 - t_1 + t_2$. Hence $(H_1 \cap B)(H_2 \cap B)$ is the set of all elements of B of the form a^f where f is in the ideal $\mathcal{K}$ in $\mathbb{Z}T$ generated by $1 - t_1^2 + t_1t_2$ and $1 - t_1 + t_2$. Since $d_1 = d_2 = a^{t_1-1}$, it remains to show that $t_1 - 1$ is in $\mathcal{K}$. But this is clear since

$$t_1 - 1 = t_1(1 - t_1 + t_2) - (1 - t_1^2 + t_1t_2).$$

This completes the proof that $H_1 \cap H_2$ is finitely generated.

Next we show that $H_1 \not\leq H_2$ by showing that $1 - t_1^2 + t_1 t_2 \notin \mathcal{J} = (1 - t_1 + t_2)$.

$$\begin{aligned} 1 - t_1^2 + t_1 t_2 \in \mathcal{J} &\Leftrightarrow t_1^{-1} - t_1 + t_2 \in \mathcal{J} \\ &\Leftrightarrow t_1^{-1} - 1 \in \mathcal{J} \\ &\Leftrightarrow 1 - t_1 \in \mathcal{J} \\ &\Leftrightarrow t_2 \in \mathcal{J} \\ &\Leftrightarrow \mathcal{J} = \mathbb{Z}T, \end{aligned}$$

but this is the case if and only if $1 - t_1 + t_2$ is invertible in $\mathbb{Z}T$. But by [**?**], the invertible elements of $\mathbb{Z}T$ are all of the form $\pm t$ for some $t \in T$.

Next we show that $H_2 \not\leq H_1$ by showing that $1 - t_1 + t_2 \notin \mathcal{I} = (1 - t_1^2 + t_1 t_2)$.

$$\begin{aligned} 1 - t_1 + t_2 \in \mathcal{I} &\Leftrightarrow t_1 - t_1^2 + t_1 t_2 \in \mathcal{I} \\ &\Leftrightarrow t_1 - 1 \in \mathcal{I}. \end{aligned}$$

Therefore, if $1 - t_1 + t_2 \in \mathcal{I}$ then $t_2 \in \mathcal{I}$, so $\mathcal{I} = \mathbb{Z}T$. Since $1 - t_1 + t_2$ is not invertible in $\mathbb{Z}T$, this is impossible. This completes the proof of the proposition. □

A final example: Finally we construct an example which is of interest from several perspectives. Let $H = \langle\, a \,\rangle \text{ wr } \langle\, s \,\rangle$ be the wreath product of the cyclic group generated by a with the cyclic group generated by s. If we introduce the abbreviation $a_i = a^{s^i}$ then H can also be viewed as the split extension of the free abelian group B with basis $\{a_i \ (i \in \mathbb{Z})\}$ by the infinite cycle on s which acts by $a_i \mapsto a_{i+1}$. Also H has a presentation as $H = \langle\, a, s \mid [a, a^{s^i}] = 1, i \in \mathbb{Z} \,\rangle$ and is finitely generated but not finitely presented.

Now we define G to be the ascending HNN extension of H by the stable letter t which acts by $a \mapsto a^2$ and $s \mapsto s$. Thus

$$\begin{aligned} G &= \langle\, H, t \mid t^{-1} s t = s, t^{-1} a t = a^2 \,\rangle \\ &= \langle\, a, s, t \mid [a, a^{s^i}] = 1, t^{-1} s t = s, t^{-1} a t = a^2 \,\rangle. \end{aligned}$$

Then G is also finitely generated but not finitely presented. Notice that the subgroup generated by $\{a, t\}$ has the familiar presentation $BS(1,2) = \langle\, a, t \mid t^{-1} a t = a^2 \,\rangle$ and has derived group isomorphic to the additive group of $\mathbb{Z}[\frac{1}{2}]$. It follows that the the derived group $[G, G]$ is the normal closure of the element a and is isomorphic to the direct sum of countably many copies of $\mathbb{Z}[\frac{1}{2}]$ which are conjugate by the action of s.

We next consider the subgroup K of G generated by the two elements $\{a, st\}$. If we abbreviate $u = st$ then we calculate $u^{-k}a^m u^k = t^{-k}s^{-k}a^m s^k t^k = t^{-k}a_k^m t^k$. In case $k \geq 0$ this becomes $u^{-k}a^m u^k = a_k^{m2^k}$. But if $k < 0$ the t^k or equivalently u^k can only be pinched when $m = n2^{|k|}$ for some n, and then $u^{-k}a^{n2^{|k|}}u^k = a_k^n$. Consequently the elements of K which are equal to u-free words lie in the subgroup of B generated by the elements $\{\dots, a_{-2}, a_{-1}, a_0, a_1^2, a_2^4, a_3^8, \dots\}$.

Now observe the words of H which represent elements of K must have exponent sum 0 on s and hence lie in B. It follows that $H \cap K$ is exactly the subgroup of B which we have just described, that is,

$$H \cap K = \langle\ \dots, a_{-2}, a_{-1}, a_0, a_1^2, a_2^4, a_3^8, \dots\ \rangle.$$

Of course $G/[G,G]$ is the free abelian group on $\{s, t\}$. $H \cap K$ is closed under the action of positive powers of t and of u but not negative powers. It is also closed under negative powers of s but not positive powers. More generally $H \cap K$ is not a submodule of $[G,G]$ for any non-trivial subgroup of $G/[G,G]$. So $H \cap K$ is the intersection of two finitely generated groups but is not finitely hybrid–presentable as a subgroup. Consequently not all of our results carry over to finitely generated metabelian groups in general. Notice however that $H \cap K$ is finitely generated by $\{a_0\}$ as a module over the polynomial ring $\mathbb{Z}[u, s^{-1}]$ with monoid generators $\{u, s^{-1}\}$. This suggests the notion of hybrid–presentable may need to be expanded to include such modules.

Since this group is generated by the subgroups H and K, it follows that the subgroup $[H,K]$ is normal in G. Now $[a, s] = a_0^{-1}a_1$ and $[a, u] = a_0^{-1}a_1^2$ both lie in $[H,K]$ and so $a_1 = a^s \in [H,K]$. Since $[H,K]$ is normal, it follows that $[H,K] = [G,G]$. One can also check that $[H,K]/(H \cap K)$ is isomorphic to the direct sum of countably many copies of the group of 2^n-th roots of unity.

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